The Basic Writings of Josiah Royce, Volume II

McDermott, John J.

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§ 15. When the methodical procedure of any more exact physical science has led to success, the result is one which the well known definition that Kirchhoff gave of the science of Mechanics exemplifies. The facts of such a science, namely, are "described" with a certain completeness, and in as "simple," that is, in as orderly a fashion as possible. The types of order used in such a description are at once "forms of thought," as we shall soon see when we enumerate them, and forms of the world of our physical experiences in so far, but only in so far as, "approximatively" and "probably," our descriptions of the world of the facts of "possible physical experience" in these terms are accurate. The philosophical problem as to how and why the given facts of physical experience conform as nearly as they do to the forms of our thought, is a question that can be fairly considered only when the types of order themselves have been discussed precisely as forms of thought, that is as "constructions" or "inventions," or "creations," or otherwise stated, as "logical entities," which our processes of thinking can either be said to "construct" or else be said to "find" when we consider, not the physical, but the logical realm itself, studying the order-types without regard to the question whether or no the physical world exemplifies them.

[Reprinted from PrL., secs. 2 and 3, pp. 92-135.]
That this mode of procedure, namely the study of the order-types apart from our physical experience, is important for our whole understanding of our logical situation (as beings whose scientific or thoughtful interpretation of nature is in question), is especially shown by the considerations with which our sketch of Methodology has just closed. For it is notable that all highly developed scientific theories make use of concepts,—such for instance as the quantitative concepts,—whose logical exactness is of a grade that simply defies absolutely precise verification in physical terms. The Newtonian theory of gravitation, for instance, can never be precisely verified. For the conception of a force varying inversely with the square of the distance, with its use of the concept of a material particle, involves consequences whose precise computation (even if the theory itself did not also involve the well known, still insurmountable, deductive difficulties of the problem of the gravitative behaviour of three or more mutually attracting bodies), would result in the definition of physical quantities that, according to the theory, would have to be expressed, in general, by irrational numbers. But actual physical measurements can never even appear to verify any values but those expressed in rational numbers. Theory, in a word, demands, in such cases, an absolute precision in the definition of certain ideal entities. Measurement, in its empirical sense, never is otherwise than an approximation, and at best, when absolutely compared with the ideal, a rough one.

Why such concepts, which can never be shown to represent with exactness any physical fact, are nevertheless of such value for physical science, our methodological study has now shown us. Their very unverifiability, as exactly defined concepts about the physical world, is the source of their fecundity as guides to approximate physical verification. For what the observers verify are the detailed, even if but approximate correspondences between very large samples of empirical data, and samples of the consequences of hypotheses. The exactness of the theoretical concepts enables the consequences of hypotheses to be computed, that is, deductively predetermined, with a wealth and variety which far transcend precise physical verification, but which, for that very reason, constantly call for and anticipate larger and larger samples of facts of experience such as can furnish the relative and approximate verifications. It is with theoretical science as it is with con-
duct. The more unattainable the ideals by which it is rationally

guided, the more work can be done to bring what we so far possess
or control into conformity with the ideal.

The order-systems, viewed as ideals that our thought at once,
in a sense "creates," and, in a sense "finds" as the facts or "entities"
of a purely logical (and not of a physical) world, are therefore
to be studied with a true understanding, only when one considers
them in abstraction from the "probable" and "approximate" ex-
emplifications which they get in the physical world.

§ 16. Yet the logician also, in considering his order-types, is
not abstracting from all experience. His world too is, in a perfectly
genuine sense, empirical. We have intentionally used ambiguous
language in speaking of his facts as either his "creations" or his
"data." For if we say that, in one sense, he seems to "create" his
order-types (just as Dedekind, for instance, calls the whole num-
bers "freie Schöpfungen des menschlichen Geistes"), his so-called
"creation" is, in this case, an experience of the way in which his
own rational will, when he thinks, expresses itself. His so-called
"creation" of his order-types is in fact a finding of the forms that
characterize all orderly activity, just in so far as it is orderly,
and is therefore no capricious creation of his private and personal whim
or desire. In his study of the Science of Order, the logician experi-
ences the fact that these forms are present in his logical world, and
constitute it, just because they are, in fact, the forms of all rational
activity. This synthetic union of "creation" and "discovery" is, as
we shall see, the central character of the world of the "Pure
Forms."

A survey of the forms of order may therefore well begin by
viewing them empirically, as a set of phenomena presented to the
logician by the experience which the theoretical or deductive
aspect of science furnishes to any one who considers what human
thought has done. The most notable source of such an experience
is of course furnished by the realm of the mathematical sciences,
whose general business it is to draw exact deductive conclusions
from any set of sufficiently precise hypotheses. If one considers
the work of Mathematics,—analyzing that work as, for instance,
the Italian school of Peano and his fellow workers have in recent
years been doing,—one finds that the various Mathematical Sci-
ences use certain fundamental concepts and order-systems, and
that they depend for their results upon the properties of these concepts and order-systems. Let us next simply report, in an outline sketch, what some of these concepts and systems are.

§ 17. Relations. One "concept," one "logical entity," or (to use Mr. Bertrand Russell's term, employed in his Principles of Mathematics) one "logical constant," which is of the utmost importance in the whole Theory of Order, is expressed by the term Relation. Without this concept we can make no advance in the subject. Yet there is no way of defining this term relation without using other terms that, in their turn, must presuppose for their definition a knowledge of what a relation is. In order, then, not endlessly to wait outside the gate of the Science of Order, for some "presuppositionless" concept that can show us the way in, we may well begin with some observation that can help us to grasp what is meant when we speak of a relation. A formal definition "without presuppositions" is impossible, whenever we deal with any terms that are of fundamental significance in philosophy.

Any object, physical or psychical or logical, whereof we can think at all, possesses characters, traits, features, whereby we distinguish it from other objects. Of these characters, some are qualities, such as we ordinarily express by adjectives. Examples are hard, sweet, bitter, etc. These qualities, as we usually conceive of them, often seem to belong to their object without explicit reference to other objects. At all events they may be so viewed. When we think of qualities, as such, we abstract from other things than the possessors of the qualities, and the qualities themselves. But, in contrast with qualities, the relations in which any object stands are characters that are viewed as belonging to it when it is considered with explicit reference to, that is, as in ideal or real company with another object, or with several other objects. To be viewed as a father is to be viewed with explicit reference to a child of whom one is father. To be an equal is to possess a character that belongs to an object only when it exists along with another object to which it is equal; and so on.

In brief, a relation is a character that an object possesses as a member of a collection (a pair, a triad, an n-ad, a club, a family, a nation, etc.), and which (as one may conceive), would not belong to that object, were it not such a member. One can extend this definition from any one object to any set of objects by saying that a relation is a character belonging to such a set when the
members of the set are either taken together, or are considered along with the members of still other sets.

It is often assumed that relations are essentially dyadic in their nature; that is, are characters which belong to a member of a pair as such a member, or to the pair itself as a pair. The relation of a father, or that of an equal, or that of a pair of equals, may be viewed as such a dyadic relation. But, as fact, there are countless relations which are triadic, tetradic,—polyadic, in any possible way. When, for instance, is an object a gift? When, and only when there exists the triad: giver, person or other entity whereunto something is given, and object given. When is an object a legal debt? Only, in general, when creditor, debtor, debt, and consideration or other ground for which or by virtue of which the debt has been incurred, exist. So that the debtor-relation: “a owes b, to c, for d,” is in general a tetradic relation. Relations involving still more numerous related objects or terms are frequent throughout the exact sciences.

If a relation is dyadic, we can readily express the proposition which asserts this relation by using the symbol \((a \, R \, b)\), meaning: “The entity \(a\) stands in the relation \(R\) to \(b\).” Whenever the proposition \((a \, R \, b)\) is true, there is always also a relation, often symbolized by \(R^\prime\), in which \(b\) stands to \(a\). This may be called the inverse relation of the relation \(R\). Thus if: “\(a\) is father of \(b\),” “\(b\) is child of \(a\),” and if one hereby means “child of a father” the relation child of is, in so far, the inverse of the relation father of.

If a relation is polyadic, then such symbols as \(R(a \, b \, c \, d \ldots)\), meaning “\(a, b, c, d, \ldots\)”, (taken in a determinate order or way which indicates the place of each in the relational \(n\)-ad in question), stand in the (polyadic) relation \(R\).” Thus, with due definition of terms \(R \, (a \, b \, c \, d)\) may be used to symbolize the assertion: “\(a\) owes \(b\) to \(c\) for (or in consideration of) \(d\);” and so on.

§ 18. Logical Properties of Relations. Relations are of such importance as they are for the theory of order, mainly because, in certain cases, they are subject to exact laws which permit of a wide range of deductive inference. To some of these laws attention must be at once directed. They enable us to classify relations according to various logical properties. Upon such properties of relations all deductive science depends. The doctrine of the Norms of deductive reasoning is simply the doctrine of these relational properties when they are viewed as lawful characteristics of rela-
tions which can guide us in making inferences, and thus Logic as the “Normative Science” of deductive inference is merely an incidental part of the Theory of Order.

Dyadic relations may be classified, first, as Symmetrical and Non-symmetrical relations. A symmetrical dyadic relation is sometimes defined as one that is identical with its own inverse relation. Or again, if $S$ is a symmetrical relation, then, whenever the assertion $(a S b)$ is true, the assertion $(b S a)$ is true, whatever objects $a$ and $b$ may be. The relation of equality, symbolized by $=$, is a relation of this nature, for if $(a = b)$, then always $(b = a)$.

If a relation is non-symmetrical, various possibilities are still open. Thus, if $R$ be a non-symmetrical relation, and if $(c R d)$, the relation $R$ may be such that the assertion $(d R c)$ is always excluded by the proposition $(c R d)$, so that both cannot be true at one of whatever $(c, d)$ one may use as the “terms” of the relation, then, in this case, the relation $R$ is totally non-symmetrical. Russell proposes to call such relation Asymmetrical. The relation “greater than” is of this type in the world of quantities. But in other cases the relation $R$ may be such that $(c R d)$ does not exclude $(d R c)$ in every instance, but only in certain instances. In the case of different relations, the exceptional instances may be for a given $R$, unique, or may be many, and may be in certain cases determined by precise subordinate laws of their own. Thus it may be the law that $(c R d)$ excludes $(d R c)$, unless some other relational proposition $(e R' f)$ is true; while if $(e R' f)$ is true, then $(c R' d)$ necessitates $(d R c)$; and so on.

Without reference to the foregoing concept of symmetry, the dyadic relations may be classified afresh, by another and independent principle, which divides them into Transitive and Non-Transitive relations. This new division is based upon considerations which arise when we consider various pairs of objects with reference to some one relation $R$. If, in particular, $(a R b)$ and $(b R c)$, the relation $R$ may be such that $(a R c)$ is, under the supposed conditions always true, whatever the objects $(a, b, c)$ may be, then in this case the relation $R$ is transitive. If such a law does not universally hold, the relation $R$ is non-transitive. The relation, equal to, is a transitive relation, according to all the various definitions of equality which are used in the different exact sciences. The so-called “axiom” that “Things equal to the same thing are equal to each other” is, in fact, a somewhat awkward expression.
of this transitivity, which, by definition, is always assigned, in any
exact science to the relation $\equiv$. The expression is awkward, be-
cause, by the use of “each other” in the so-called “axiom,” the
transitivity of the relation $\equiv$, is so stated as not to be clearly
distinguished from the symmetry which also belongs to the same
relation. Yet transitivity and symmetry are mutually independent
relational characters. The relations, “greater than,” “superior to,”
etc., are, like the relation $\equiv$, transitive, but they are totally non-
symmetrical. The relations “opposed to,” and “contradictory of”
are both of them symmetrical, but are also non-transitive.

Fewer formulations of this general type have done more to
confuse untrained minds than the familiar “axiom”: “Things
equal to the same thing are equal to each other,” because the form
of expression used suggests that the relation, $\equiv$, possesses its transi-
tivity because of its symmetry. Everybody easily feels the sym-
metry of the relation $\equiv$. Everyone admits (although usually
without knowing whether the matter is one of definition, or is one
of some objectively necessary law of reality, true apart from our
definitions), that the relation $\equiv$ is transitive. The “axiom” suggests
by its mode of expression that this symmetry and this transitivity
are at least in this case, necessarily united. The result is a wide-
spread impression that the symmetry of a relation always implies
some sort of transitivity of this same relation.—an impression which
has occasionally appeared in philosophical discussions. But no-
where is a sharp distinction between two characters more needed
than when we are to conceive them as, in some special type of
cases necessarily united, whether by arbitrary definition or by the
nature of things.

If some dyadic relation, say $X$, is non-transitive, then there is
at least one instance in which the propositions $(d X e)$ and $(e X f)$
are both of them true of some objects $(d, e, f)$, while $(d X f)$ is
false. As in the case of the non-symmetrical relations, so in the
case of the non-transitive relations, this non-transitivity, like the
before mentioned non-symmetry, may appear in the form of an
universal law, forbidding for a given relation $R$ all transitivity; or
else in the form of one or more special cases where a given relation
does not conform to the law that the principle of transitivity
would require. These special cases may be themselves subject to
special laws. A relation, $T$, is totally non-transitive, in case the two
assertions $(a T b)$ and $(b T c)$ if both at once true, exclude the
possibility that \((a \lor c)\) is true. Thus if \(a\) is father of \(b\) and \(b\) is father of \(c\), it is impossible that \(a\) is father of \(c\) should be true.

The relation father of, is both totally non-symmetrical and also totally non-transitive. That relation between propositions which is expressed by the verb "contradicts," or by the expression "is contradictory of," is symmetrical, but totally non-transitive. For propositions which contradict the same proposition are mutually equivalent propositions. The relation "greater than," as we have seen, is transitive, but totally non-symmetrical. The relation \(=\) is both transitive and symmetrical. And thus the mutual independence of transitivity and symmetry, as relational properties, becomes sufficiently obvious.

Still a third, and again an independent classification of dyadic relations appears, when we consider the number of objects to which one of two related terms can stand, or does stand, in a given relation \(R\), or in the inverse relation \(R\). If \("a\) is father of \(b\)," it is possible and frequent that there should be several other beings, \(e, d,\) etc., to whom \(a\) is also father. If \("m\) is twin-brother of \(n,"\) then, by the very definition of the relation, there is but one being, \(viz. n,\) to whom \(m\) can stand in this relation. If \("e\) is child of \(f,"\) there are two beings, namely the father and the mother, to whom \(e\) stands in this relation. In a case where the estate of an insolvent debtor is to be settled, and where the debtor is a single person (not a partnership nor yet a corporation), then the transactions to be considered in this one settlement may involve many creditors, but, by hypothesis, only one debtor, so far as this insolvent's estate alone is in question. Here, there are then several beings, \((p, q, r,\) etc.), of each of whom the assertion can be made: \(-"p\) is creditor of \(x."\) But so far as this one case of insolvency alone is concerned, all the creditors in question are viewed as a many to whom only one debtor corresponds, as the debtor here in question.

The questions suggested by such cases are obviously capable of very variously multiplex answers, according to the relational systems concerned. Of most importance are the instances where some general law characterizes a given relation \(R\), in such wise that such questions as the foregoing cases raise can be answered in universal terms. The principal forms which such laws can take are sufficiently indicated by the three following classes of cases:--

1. The relation \(R\) may be such that, if \((a \lor b)\) is true of some pair of individual objects \((a, b)\), then, in case we consider one of
these objects, \( b \), there are or are possible other objects, besides \( a \),—objects \( m, n, \) etc.,—of which the assertions \((m \ R \ b)\), \((n \ R \ b)\), etc., are true; while at the same time, if we fix our attention upon the other member of the pair, \( a \), there are other objects \((p, q, r)\) either actual, or, from the nature of the relation \( R \), possible, such that \((a \ R \ p)\), \((a \ R \ q)\), etc., are true propositions. Such a relation \( R \) is called by Russell and others a "many-many" relation. The laws that make it such may be more or less exact, general and important. Thus the relation "\( r^o \) of latitude south of" is such a "many-many" relation, subject to exact general laws.

2. The relation \( R \) may be such that, when \((a \ R \ b)\) is true of some pair \((a, b)\), the selection of \( a \) is uniquely determined by the selection of \( b \) while, given \( a \), then, in place of \( b \), any one of some more or less precisely determined set of objects could be placed. Thus if "\( a \) is sovereign of \( b \)" where the pair \((a, b)\) is a pair of persons, and where the relation sovereign of is that of some one wholly independent kingdom (whose king's sovereign rights are untramelled by feudal or federal or imperial relationships to other sovereigns),—then, by law, there is only one \( a \) whereof the assertion: "\( a \) is sovereign of \( b \)" is true. But if we first choose \( a \), there will be many beings that could be chosen in place of \( b \), without altering the truth of the assertion. A case of such a relation in the exact sciences is the case "\( a \) is centre of the circle \( b \)." Here, given the circle \( b \), its centre is uniquely determined. But any one point may be the centre of any one of an infinite number of circles. Such a relation \( R \) is called a "one-many" relation. Its inverse \( R \) would be called a "many-one" relation.

3. A relation \( R \) may be such that (whether or no there are many different pairs that exemplify it), in case \((a \ R \ b)\) is true of any pair whatever, the selection of \( a \) uniquely determines what one \( b \) it is of which \((a \ R \ b)\) is true, while the selection of \( b \) uniquely determines what \( a \) it is of which \((a \ R \ b)\) is true. Such a relation is called a "one-one" relation. Couturat prefers the name "bi-univocal" relation in this case. The "one-one" relations, or, as they are often called "one-one correspondences," are of inestimable value in the order systems of the exact sciences. They make possible extremely important deductive inferences, for example those upon which a great part of the modern "Theory of Assemblages" depends.

The various classifications of dyadic relationships that have now been defined, may be applied, with suitable modifications, to tri-
adic, tetradic, and other polyadic relations. Only, as the sets of
related terms are increased, the possible classifications become, in
general, more varied and complicated. A few remarks must here
suffice to indicate the way in which such classifications of the
polyadic relations would be possible.

If the symbol $S (a\ b\ c\ d\ .\ .\ .)$ means: “The objects $a$, $b$, $c$, $d$,
etc., stand in the symmetrical polyadic relation $S$,” then the objects
in question can be mutually substituted one for another, i.e. the
symbols $a$, $b$, $c$, etc., can be interchanged in the foregoing expres­
sion, without altering the relation that is in question, and without
affecting the truth of the assertion in question. This is for instance
the case if $S (a\ b\ c\ d\ .\ .\ .)$ means: “$a$, $b$, $c$, $d$, . . . are fellow-members
of a certain club,” or: “are points on the same straight line,” so
long as no other relation of the “fellow-members” or of the
“points” is in question except the one thus asserted. In such cases
$S (a\ b\ c\ d\ .\ .\ .)$, $S (b\ c\ d\ a\ .\ .\ .)$, etc., are equivalent propositions.
Such a relation $S$ is polyadic and symmetrical. The relation $R$,
expressed by the symbol $R (a\ b\ c\ d)$, is non-symmetrical (partially
or totally) if in one, in many, or in all cases where this relation is
thus asserted there is some interchanging of the terms or of the
objects,—some substituting of one for another,—which is not per­
mitted without an alteration of the relation $R$, or a possible destruc­
tion of the truth of the relational proposition first asserted. This is
the case if $R (a\ b\ c\ d)$ means: “$a$ owes $b$ to $c$ for, or in considera­
tion of $d$;” or, in a special case “$a$ owes ten dollars to $c$ for one
week’s wages.” Such a relation is non-symmetrical. The number
of terms used greatly increases the range of possibilities regarding
what sorts of non-symmetry are each time in question; since, in
some cases, certain of the terms of a given polyadic relational
assertion can be interchanged, while others cannot be interchanged
without an alteration of meaning or the change of a true into a
false assertion. Thus if the assertion $R (a\ b\ c\ d)$ means “$a$ and $b$ are
points lying on a certain segment of a straight line whose extremi­
ties are $c$ and $d$,”—then $a$ and $b$ can be interchanged, and $c$ and $d$
can be interchanged, without altering the truth or falsity of the
assertion; but if the pair $(a, b)$ is substituted for the pair $(c, d)$, and
conversely, the assertion would in general be changed in its mean­
ing, and might be true in one form, but false when the inter­
change was made. Consequently we have to say, in general, that
a given polyadic relation, $R$, is symmetrical or non-symmetrical
with reference to this or that pair or triad or other partial set of its terms, or with reference to this or that pair of pairs, or pair of triads, of its terms; and so on. In case of complicated order-systems, such as those of functions in various branches of mathematics, or of sets of points, of lines, etc., in geometry, the resulting complications may be at once extremely exact and definable, and very elaborate, and may permit most notable systems of deductive inferences.

In place of the more elementary concept of transitivity, a more general, but at the same time more plastic concept, in terms of which certain properties of polyadic relations can be defined, is suggested by the process of elimination, so familiar in the deductive inferences of the mathematical sciences. Suppose \( R(a b c d) \) is a tetradic relation, symmetrical or non-symmetrical; suppose that the relation is such that if the propositions \( R(a b c d) \) and \( R(c d e f) \) are at once true, then \( R(a b e f) \) necessarily follows.

A very notable instance of such a relation exists in the case of the "entities of Pure Logic" of which we shall speak later. We could here easily generalize the concept of transitivity so as to say that this relation \( R \) is "transitive by pairs." But such transitivity, as well as the transitivity of a dyadic relation, is a special instance of a general relational property which permits the elimination of certain terms that are common to two or more relational propositions, in such wise that a determinate relational proposition concerning the remaining terms can be asserted to be true in case the propositions with which we began are true. Let the symbol \( a \) represent, not necessarily a single object, but any determinate pair, triad, or \( n \)-ad of objects. Let \( \beta \) represent another such determinate set of objects, and \( \gamma \) a third set. Let \( R \) and \( R' \) be polyadic relations such that \( R(a \beta) \) and \( R'(\beta \gamma) \). The first of these symbols means the assertion: "The set of objects consisting of the combination of the sets \( a \) and \( \beta \) (taken in some determinate mode or sequence), is a set of objects standing in the relation \( R \)." The second symbol, viz. \( R'(\beta \gamma) \) is to be interpreted in an analogous way. Hereupon, suppose that either always, or in some definable set of cases, the proposition \( R(a \beta) \) and \( R'(\beta \gamma) \), if true together, imply that \( R''(a \gamma) \), where \( R'' \) is some third polyadic relation, which may be, upon occasion, identical with either or both of the foregoing relations, \( R' \) and \( R \). In such a case, the information expressed in \( R(a \beta) \) and \( R'(\beta \gamma) \), is such as to permit the elimina-
tion of the set or collection \( \beta \), so that a determinate relational proposition, results from this elimination. It is plain that transitivity, as above defined, is a special instance where such an elimination is possible.¹

With regard to the "one-one," "many-one" and "many-many" classification of dyadic relations, we may here finally point out that a vast range for generalizations and variations of the concepts in question is presented, in case of triadic, and, in general, of polyadic relations, by the "operations" of the exact sciences—operations which have their numerous more or less "approximate" analogues in the realm of ordinary experience. These operations make possible deductive inferences whose range of application is inexhaustible.

An "operation," such as "addition" or "multiplication," is (in the most familiar cases that are used in the exact sciences) founded upon a triadic relation. If \( R(a, b, c) \) means "The sum of \( a \) and \( b \) is \( c \)," or in the usual symbolic form, \( a + b = c \), then the triadic relation in question is that of two numbers or quantities to a third number or quantity called their "sum." As is well known, the choice of two of these elements, namely the choice of the \( a \) and \( b \) that are to be added together (the "summands"), determines \( c \) uniquely, in ordinary addition. That is, to the pair \( (a, b) \) the third element of the triad \( (a, b, c) \) uniquely corresponds, in case \( R(a, b, c) \) is to be true. On the other hand, given \( c \), the "sum," there are in general, various, often infinitely numerous, pairs \( (d, e) \), \( (f, g) \), etc., of which the propositions, \( d + c = c \), \( f + g = c \), etc., may be true. But in case of ordinary addition if \( c \), the "sum," is first given, and if then one of the "summands," say \( a \), is given, the other, say \( b \), can always be found (if the use of "negative"

¹ In the closing chapter of his Psychology, in a beautiful sketch of the psychological aspects of scientific thinking, Professor Wm. James characterizes the transitivity of those dyadic relations, which are so often used in the natural sciences, by saying that the objects whose relations are of this transitive type follow what he calls "The axiom of skipped intermediaries." This is a characteristically concrete way of stating the fact that one main deductive use of transitivity, as a relational property, lies in the fact that it permits certain familiar eliminations. If, namely: "\( a \) is greater than \( b \), and \( b \) is greater than \( c \)," we may eliminate the intermediary \( b \), and conclude deductively that \( a \) is greater than \( c \). We are here concerned, in our text, with the fact that dyadic transitivity is only a special instance of the conditions that make elimination in general possible, and that determine a whole class of Norms of deductive inference.
numbers or quantities is indeed permitted in the system with which
we are dealing), and, when found, is then uniquely determined.
Triadic relations, such as that which characterizes addition, may
therefore be subject to precise laws whereby, to one element, or
to two elements which are to enter into a triad, either one or many
ways of completing the triad may correspond, these possible
ways varying with the relational proposition whose truth is to be,
in a given case, asserted or denied, or is to remain unchanged
through the substitution of various new objects for those already
present in a given triad.

The "operations" of the exact sciences are of inestimable impor­tance for all the order-systems in terms of which precise theories
are defined and facts are described. It is not necessary that they
should precisely resemble, in their relational properties, either the
"multiplication" or the "addition" of the ordinary numbers and
quantities. A glance at their possible varieties (as these are discussed
in connection with modern "group-theory," or as a part of the
treatment of the various "algebras" which newer mathematics has
frequent occasion to develop), will readily show to any thoughtful
observer the absurdity of the popular opinion, still often enter­tained by certain philosophical students, that "mathematics is the
science of quantity." The "quantities" are objects that are indeed
vastly important. Their "order-system" is definable in terms of a
few important properties of certain dyadic and triadic relations.

All our power to reason deductively about quantity depends upon
these few relational properties, whose consequences are neverthe­less inexhaustibly wealthy. But the algebra of quantity is one only
of infinitely numerous algebras whose operations are definable in
terms of triadic relations. And there is no reason why other
operations should not be defined in terms of tetradic, and, in fact
of n-adic relations. The "Algebra of Pure Logic" is, in fact, as
Mr. Kempe has shown, the symbolism of a system whose "opera­tions" are superficially viewed, triadic, but are really founded
upon tetradic relations (see § 24, below). And mathematical sci­ence includes within its scope the deductive reasonings possible
in case of all these order-systems, and capable of being symbolized
by all these algebras.

§ 19. Classes. In describing relations and their properties, we
have inevitably presupposed the familiar concept of a set or collec­tion, i.e. of a class of objects as already known. Relations are
impossible unless there are also classes. Yet if we attempt to define this latter concept, we can do so only by presupposing the conception of Relation as one already understood. As we have already pointed out, such a “circle in definition” is inevitable in dealing with all philosophical concepts of a fundamental nature.

The concept of a Class or Set or Collection or Assemblage (Menge) of objects, is at once one of the most elementary and one of the most complex and difficult of human constructions. The apparent commonplaces of the Socratic-Platonic Methodology, and their intimate relation to the profound problems of the Platonic Metaphysics, which we touched upon in § 3, have shown us from the outset how the most obvious and the deepest considerations are united in this problem. The “burning questions” of the new “Theory of Assemblages” as they appear in the latest logical-mathematical investigations of our days, illustrate surprisingly novel aspects of the same ancient topic.

The concept of a Class, in the logical sense, depends (1) Upon the concept of an Object, or Element or Individual, which does or does not belong to a given class; (2) Upon the concept of the relation of belonging to, i.e., being a member of a class, or of not so belonging; (3) Upon the concept of assertions, true or false, which declare that an object is or is not a member of a given class; (4) Upon the concept of a Principle, Norm, or Universal which enables us to decide which of these assertions are true and which are false.

The first of these concepts is in many ways the most problematic of all the concepts used in the exact sciences. What constitutes an Individual, what is the “principle of individuation,” how are individuals known to exist at all, how are they related to universal types, how they can be identified in our investigations, or how they can be distinguished from one another, whether they can be “numerically distinct” and yet wholly or partially similar or identical,—these are central problems of philosophy, which we in vain endeavour to escape by asserting in the usual way that “individuals are presented to us as empirical objects, by our senses.” Whoever has had occasion to study any problem involving the doubtful or disputed identity of any individual object, knows that no direct sense-experience ever merely presents to us an individual object such as we conceive of, where we subject our processes of identification to exact rules and tests.
For logical purposes, an Individual Object is one that we propose to regard at once as recognizable or identifiable throughout some process of investigation, and as unique within the range of that investigation, so that no other instance of any mere kind of object suggested by experience, can take the precise place of any one individual, when we view ourselves as having found any individual object. Thus to propose to treat an object as always recognizable under certain conditions, and as such that no substitute for it is possible, in so far as we treat it as this individual,—all this involves an attitude of will which our sense-experience can illustrate and more or less sustain, but can never prove to be necessary, or present to us as successfully and finally warranted by mere data.

The concept of an individual is thus one whose origin and meaning are due to our will, to our interest, to so-called pragmatic motives. We actively postulate individuals and individuality. We do not merely find them. Yet this does not mean that the motives which guide our will in this postulate are wholly arbitrary, or are of merely relative value. There are some active and voluntary attitudes towards our experience which we cannot refuse to take without depriving ourselves of the power to conceive any order whatever as present in our world. Without objects conceived as unique individuals, we can have no Classes. Without classes we can, as we have seen, define no Relations, without relations we can have no Order. But to be reasonable is to conceive of order-systems, real or ideal. Therefore, we have an absolute logical need to conceive of individual objects as the elements of our ideal order systems. This postulate is the condition of defining clearly any theoretical conception whatever. The further metaphysical aspects of the concept of an individual we may here ignore. To conceive of individual objects is a necessary presupposition of all orderly activity.

An individual once postulated as present may be classed with other individuals. If the various individuals in question are viewed as if they were already given, the act of classing them thus, that is of asserting that these individuals belong in the same class, is again an act of will. Its value is so far pragmatic. We accomplish in this way some purpose of our own, some purpose of treating things as for some special reason distinguished or, on the other hand, undistinguished. In this sense, all classes are subjectively distinguished from other classes by the voluntarily selected Norms, or principles of classification which we use. Apart from some classify-
ing will, our world contains no classes. Yet without classifications we can carry on no process of rational activity, can define no orderly realm whatever, real or ideal. In this sense, the act of defining at least some norms or principles of classification is an act whose logical value is not only pragmatic, but also absolute. For a world that we might conceive as wholly without classes, would be simply no world at all. We could do nothing with it or in it. For to act, consciously and voluntarily, in any way whatever is to classify individuals into the objects that do and into those that do not concern, meet, serve, correspond to, stimulate or result from each sort of activity. Thus classes are in one sense "creations," in another sense absolute presuppositions of all our voluntary activity, and so of all our theories.

If we have in mind some norm of principle of classification, this norm inevitably defines at least one pair of classes, namely a given class and its negative or contradictory class. For if the class \( x \) is defined by a given norm, then the same norm defines the class consisting of whatever objects are not \( x \), a class here to be symbolized by \( \neg x \).

Whenever we set out to classify any region of our world, real or ideal, we of course always do so because we know, or at least postulate, that there are some individuals in that region to be classified. And considered with reference to a given norm, which defines a class \( x \), these individuals will belong either to \( x \) or else to \( \neg x \). But of course our norm does not of itself tell us whether there are any individuals, in the region to be classified, which are of the class \( x \). We can, then, define a norm for a class \( x \), and later discover that "Everything is \( x \)," so that "There are no \( x \)'s." In general, then, when we define by its norm the class \( x \), either one of two assertions may turn out to be true about \( x \). Either (1) "\( x \) has no member," or (2) "\( x \) has at least one member." Of these two assertions one is true, the other false, when uttered about any determinate class \( x \). That is, these assertions are mutually contradictory.

A very vast range of the assertions of the exact sciences can be said to be of one or the other of these two comparatively simple types. A class that has no members, a "nothing-class," an "empty class," or "zero-class" may be symbolized by \( 0 \). It is in that case a class sharply defined by its norm, but known not to contain any of the objects that we have chosen to regard or to define as the
individuals of the world (real or ideal) with which we are dealing. If a class $x$ has no members, its negative, viz. $\bar{x}$, comprises everything that belongs to the realm or (in the phrase of the English logician, De Morgan) to the "universe of discourse" with which we are dealing. The class everything can be symbolized by $I$. Regarding 0 and 1 as classes, and using $\equiv$ as the symbol, in the present case, of the relation of logical equivalence or identity between any two classes, we can assert, as formally true of any world, which for any reason, we can classify, that:

\begin{align*}
(1) \quad & 0 \equiv I; \\
(2) \quad & 1 \equiv I.
\end{align*}

That is, the class nothing and the class everything are negatives each of the other, whenever these terms are used of any one "universe of discourse" into which a definite classification has been introduced.

Given any two distinct classes, $x$ and $\gamma$, defined by different norms or principles of classification, then inevitably, and without regard to whether $x$ and $\gamma$ are, either or both of them "zero," that is "empty" classes, the very definitions of $x$ and of $\gamma$ require that two new resulting classes should be present, as classes that may or may not have members, in our classified world. These new classes are: (1) The "Logical Product" of the classes $x$ and $\gamma$, that is, the class of those objects in our "universe of discourse" that conform at once to the norm of $x$ and to the norm of $\gamma$, and that, therefore belong at once to both the classes $x$ and $\gamma$; (2) The Logical Sum of the classes $x$ and $\gamma$, that is, the class of those objects that conform either to the norm of $x$ or to the norm of $\gamma$, and that therefore belong to one at least of the two classes ($x$, $\gamma$). We symbolize by $xy$ the logical product of $x$ and $\gamma$, and by $x + y$, their logical sum. In every extended discussion of classes logical sums and products are sure to occur.

Between two classes, $p$ and $q$, there may or may not exist a certain relation which is of fundamental importance for all study of classes, and so for all exact science. This is the relation of subsumption. It is a relation non-symmetrical, but not totally non-symmetrical. We may symbolize this relation by $-<$. If $p -< q$, then whatever conforms to the norm of $p$ conforms to the norm of $q$; or, as we also may say, the class $p$ is included in the class $q$. If $(p -< q)$ and $(q -< p)$ are at once true, then $(p = q)$. In case the relation $(p -< q)$ holds true, the logical product of $p$ and $\bar{q}$ has no mem-
bers, or in symbols, $p\overline{q} = \alpha$. The subsumption relation is transitive, that is:

"If $(p \prec q)$ and $(q \prec r)$ then $(p \prec r)$.”

As the modern study of the topic has shown, the entire traditional “theory of the syllogism” can be expressed as a sort of comment upon, and relatively simple application of, this transitivity of the subsumption-relation. Thus does the theory of the "norms of thought" form merely a subordinate part of the theory of Logical Order.

One relation remains here to be explicitly characterized,—a relation often confounded with the subsumption relation, but carefully distinguished therefrom, in recent times by Frege, Peano, and Russell. It is the relation in which an individual stands to the class to which it belongs, and of which it is a member. The school of Peano symbolize this relation by $\epsilon$. Thus, supposing $i$ to be the name of an individual object, the symbol $(i \epsilon x)$ means: “$i$ is a member of, that is, belongs to the class $x$.” Since a class itself can be and sometimes is treated logically as an individual, in case this class is taken as one member of a set of classes (as, for instance, when one says: “The powers of 2, such as $2^2$, $2^3$, etc., form a class that is one of the classes of whole numbers”), we can suppose the proposition $x \epsilon y$ to be true of some class $x$, $y$ being a class of classes. But in such a case, if $(i \epsilon x)$ and $(x \epsilon y)$, then the assertion $(i \epsilon y)$ is, in general, false. So that the $\epsilon$-relation is non-transitive, while the relation $\prec$, the subsumption relation is transitive. They are, then, quite different relations.

Any class, $x$, consists of the individuals, $i, i', i'' \ldots$, whereof the corresponding assertions $(i \epsilon x)$, $(i' \epsilon x)$, etc., are true. From the formal point of view it is thus possible, and in fact, for certain logical purposes necessary, to develop the “Theory of Classes” upon the basis of the “Theory of Propositions.” Propositions, themselves, have certain characteristic logical relations, of contradiction, implication, and so on. To these relations of propositions those relations of classes which we have named, viz. negation, subsumption, etc., correspond in certain exact ways. There is therefore possible a “calculus of classes”; although the two doctrines have certain notable differences regarding the principles available for deductive purposes in each of them.

The assertions of the type $(i \epsilon x)$ upon which classifications may be said to rest, have the aforesaid paradoxical character. They are,
namely, the expressions of postulates, or voluntary acts, since all classification involves a more or less arbitrary norm or principle of classification. Yet the laws to which such propositions, as well as any logical system of classes are subject, are nevertheless exact, are definable (as we have seen) in terms of precise dyadic, triadic and tetradic relations, and are not in the least arbitrary. In fact, despite the arbitrariness of each individual classification, the general laws of logic possess an absoluteness which cannot conceivably be surpassed, and lie at the basis of all order-system and of all theory.

The only possible answer to the question as to how the absoluteness of the logical principles is thus consistent with the arbitrariness of each of the classifications which we make, lies in saying that the logical principles define precisely the nature of the “Will to act in an orderly fashion” or in other words of the “Will to be rational.”

§ 20. The Types of Order. The foregoing concepts of Relation, of Relational Properties, and of Classes, have enabled modern mathematicians, and other students of logic, to define in exact terms a surprisingly vast range of order-systems. With almost dramatic suddenness the considerations which may have seemed so varied, disunited and abstract in the foregoing sketch, suddenly give us, when they are once properly combined, an insight into precisely what is most momentous about the order present in the worlds of Number, of Quantity, of Geometry, and of Theoretical Natural Science generally.

For, in the first place, what order-type is universally present wherever there is any order in the world? The answer is, Serial Order. What is a Series? Any row, array, rank, order of precedence, numerical or quantitative set of values, any straight line, any geometrical figure employing straight lines, yes all space, all time,—any such object involves serial order. Serial order may exist in two principal types, the “open” series, and the “closed” series or cycle. Since the latter type of order may be reduced to the former by certain well-known devices, it suffices here to characterize any serial order that is “open,” i.e. that does not return into itself. So viewed, a Series is:—A class of individuals or elements such that there exists a single relation \( R \), dyadic, transitive, and totally non-symmetrical, while this relation \( R \) is of such a nature that, whatever pair \((a, b)\) of distinct elements of the class in question be chosen, either \((a R b)\) or else \((b R a)\) is true. Since the relation \( R \) is
by definition totally non-symmetrical, \((a \ R \ b)\) and \((b \ R \ a)\) cannot be true at once of any chosen pair of objects belonging to the series defined in terms of \(R\). If we begin with any pair \((c, b)\), of elements of a given series, the “place” of any other element \(a\) or \(g\) is determined with reference to \(c\) and \(b\) by such assertions as \((a \ R \ c)\) and \((c \ R \ g)\), \((g \ R \ b)\), etc., while the transitivity of the relation \(R\) enables us to use such assertions as a basis for deductive inferences whenever two pairs with a common element appear in the course of our determinations. Chains of inference, eliminations, etc., result. Thus, once more, certain norms of deductive inference are determined by relational properties.

Now in terms of the variations which this definition of series permits to be present in the classes and sub-classes of which a series may consist, an infinite variety of distinct serial types can be defined upon the basis of the single definition just stated, and of the logical properties of classes.\(^2\)

The series of the positive whole numbers, for instance, is characterized by the fact that there is one member of the class in question, namely the first, which stands in a relation \(R\) to every other whole number, \(R\) being the transitive and totally non-symmetrical relation of “predecessor,” while no positive whole number stands in the relation \(R\) to this first one; and by the further fact that whatever number (say 2, or \(n\)) one chooses, there is one number (say 3, or \(n+1\)) and only one, such that, while \((n \ R \ n+1)\) is true, no whole number \(m\) exists such that \((n \ R \ m)\) while \((m \ R \ (n+1))\).

In this case \((n+1)\) is called the next successor of \(n\). And thus the relation “next successor” is defined in terms of \(R\), and of the absence of intermediates. A further characteristic of the whole numbers is this, that if any property \(Q\) belongs to the first whole number, and if \(Q\) is such that, in case \(Q\) belongs to any whole number, \(n, Q\) belongs to the “next successor” of \(n\) (say to \(n+1\)), then \(Q\) belongs to all of the whole numbers. This characteristic of the whole number series is defined and applied by combining the

\(^2\) The use of the foregoing definition, and the classifications of possible serial types which the definition permits, have now become common property. The significance of the definition, and the wealth of ordinal properties that could be stated in terms of it, were gradually brought to light in the latter half of the nineteenth century through the researches of C. S. Peirce, of Dedekind, of Cantor, and of various other logical and mathematical writers. The results have been summed up, and placed in various new lights, in Russell's *Principles of Mathematics*. 
other relational properties of the series with the logical properties of classes, and is of the most fundamental importance for deduction throughout mathematical theory. Thus still another norm of deductive reasoning is established for a certain class of cases.

Such simple considerations concerning classes and relations define then, the series of the whole numbers, and predetermine the inexhaustible wealth of the Theory of Whole Numbers. An extension of such an ordinal series “backwards” gives us the negative whole numbers. The series of the “rational numbers” can be characterized as to its ordinal type by defining the relation R, for that series, and also by choosing the elements of the series, so that whatever pair \((i, k)\) of distinct rational numbers exists, such that \((i R k)\) is true, there also exists \(j\) different from \(i\) and from \(k\) such that \((i R j)\) and \((j R k)\). A series of this type is now called “dense.”

Upon the basis of the dense series of the rational numbers we can define another series, that of the “cuts” or Schnitte of the rational numbers. This new series is (in Dedekind’s sense) “continuous.” It is defined in terms of still another union of a certain sort of classification with the relational properties already in question. This series of the “cuts” of the rational numbers is the series of the “real numbers.” And Cantor has worked out a more precise characterization of the properties of the continuous series of “real numbers” (the so-called “arithmetical continuum”) by a still further synthesis of the properties of certain sub-classes which such a series contains, with the general properties of the relation \(R\), whereby the series as a whole is determined.

In consequence, mathematical science is now in possession of a complete definition of the “arithmetical continuum” in purely ordinal terms.

But the numbers are not merely subject to dyadic ordinal relations. As usually employed in arithmetic and algebra, they are also subject to triadic relations, in terms of which the operations of ordinary addition \((a + b = c)\), and of multiplication \((pq = r)\) are defined. The momentous problem arises as to how these triadic relations are themselves related to the dyadic ordinal relations of the number-series. This problem has been attacked with complete success by the modern students of the foundations of mathematics. It has been shown, first, that the simple series of the whole numbers, defined as above, is such as to enable us to define for that series the operations of the addition and multiplication of its own
terms upon the basis of considerations that involve solely the dyadic relational properties of this whole-number series as it stands. That is, in case of a series such as the whole numbers, positive and negative, the triadic relations involved in addition and in multiplication, can be defined in terms of the dyadic relations whereby the series is ordered. But in case of the dense series of the rational numbers, and still more in case of the "arithmetical continuum" of the "real numbers," and again yet more in case of the "complex numbers" of algebra, *such a reduction of the triadic relations of these numbers to the dyadic relations of the whole numbers can be accomplished only indirectly*, by means of special definitions, which enable us to regard these other series and in fact the whole system of the "complex numbers," as derived, through a sort of "logical genesis," from the original whole number series, by a series of combinations of the terms, classes, and relations, of the latter series, and by further combinations of the results of these first combinations. All this "genesis" we have not here room to follow. It is enough to say that the result of this research is to show that all the properties which make the numbers of ordinary algebra subject to the endlessly varied operations of calculation, can be reduced to properties which depend: (1) Upon the dyadic relations of order which hold in the whole-number system itself, and (2) Upon the properties and ordinal relations of certain derived logical entities (pairs of whole numbers, classes of these pairs, pairs of real numbers, etc.). And in brief, we can say: All the properties of the numbers which are used in ordinary algebra, are properties of their order-system, while this order-system is *indirectly definable* on the basis of the properties of the whole-number system, and of the properties of certain classes and relations of objects which the whole-number system enables us to define.

The number-system of ordinary algebra being once defined, it is possible to deal, in a systematic way, with the problems which are presented by the physical and ideal *Quantities* with which mathematical theories so frequently deal. *Quantities* are objects, either physical or ideal, that fall into series by virtue of relations of the nature of *greater* and *less*. They have therefore their serial order-systems. They also, in general, are subject to relations of *equality*. In case they are *Intensive* Quantities, their order-systems are definable *only* by means of such dyadic relations, that is, by means of relations of *greater-less*, and by means of the symmetrical
relation of equality. Extensive Quantities are such as, over and above these dyadic relations, of greater, less, equal, are subject to triadic relations in terms of which the sum of any two extensive quantities that belong to the same system of quantities can be defined. In the realm of the quantities, however, there is no general mode of “logical genesis” which makes it possible for us to define triadic relations of the type \(a + b = c\) upon the sole basis of the dyadic relations greater, less, and equal. Herein the quantities differ, as logical objects, from the number-series viewed as pure algebra views the latter. The “logical genesis” of the rational and of the “real” numbers, a genesis of which we have just made mention, has no precise and general correspondent process in the world of quantities. Therefore, those triadic relations of most sets of extensive quantities upon which their addition depends, are defined, either (1) upon the basis of empirical inductions (as is the case with physical weights, with quantities of energy, etc.), or (2) upon the basis of voluntarily assumed postulates (as is the case with many systems of ideal quantities, such as for instance the extensive quantities of Pure Metrical Geometry as they are usually treated), or (3) upon some union of postulate and of physical experience (as is frequently the case in the applications of geometry, and in such a science as Mechanics). 3

Given, however, some workable and sufficiently general definition of a triadic relation upon which an addition-operation can be founded, then the number-system can be at once introduced into the theory of any system of quantities. The exactness of a physical theory of such a set of quantities depends upon such an introduction. The order-system of such a realm of extensive quantities becomes correspondent to the order-system either of a part of the numbers, or of the entire system of the real or of the complex numbers. Thus, what makes deductive inference in the realm of quantity possible depends solely upon the ordinal properties of this realm.

The application of the foregoing principles regarding serial order-types to the theory and description of more complicated

3 In the very notable case of geometrical theory, a special form of reduction of the “metrical” to the “ordinal” properties of space-forms also exists, whereby the bases of metrical geometry can be indirectly reduced to principles that are stateable wholly in projective, that is, in ordinal terms. This case is of vast importance for the logic of geometry, but cannot further be studied here.
order-systems, involves a set of processes to which we have now made frequent reference namely: The Correlation of Series. Upon such correlations the whole theory of Mathematical Functions depends—a theory which admits of infinitely numerous variations and applications, and which plays its part in every extended and exact theoretical science. The norms of deductive inference which are definable here are numerous and complex, but vastly important.

The simplest type of correlation is that which takes place when a relation of "one-one correspondence" can be established between the members of two series, or between the members of definable parts of such series. In other cases, a "one-many" relation can be established, whereby to every member (say \( p \)) of a given series \( S \), there corresponds some determinate number, a pair \((q, r)\) or a triad \((q, r, s)\) of elements, chosen from some series \( S' \), or else so that \( q \) belongs to \( S' \), \( r \) to \( S'' \), etc.; while, given \((q, r, \text{etc.})\), \( p \) is uniquely determined. The possibilities thus suggested may be still further varied without any necessary sacrifice of exactness of definition. In very numerous instances, especially where the operations possible in case of numbers and quantities are in question, we may have a correspondence and correlation of series so established that, to each set of pairs \((p, q)\), or of triads \((p, q, r)\) etc. (whereof \( p \) shall be chosen from one series, \( q \) from another or from the same series, and so on), there corresponds some determinate element \( x \), or some set of elements \((x, y, z, \text{etc.})\), while the element \( x \) (or the set \( x, y, \text{etc.} \)) can be defined as elements of some series or order-system that thus results from or that is definable in terms of the "functional relation," whose laws lie at the basis of the correlation in question. In general, let \( a, \beta, \text{and} \gamma \) be, not now single individuals viewed merely as such, but pairs, triads, or other sets of objects. Let the elements whereof each of the sets \( a, \beta, \text{and} \gamma \), consists, be all chosen in a determinate way from certain series of objects already defined (number-series, points on lines, series of lines or of other geometrical figures, physical quantities, etc.). Suppose that some general law exists which one can state in the form: "If \( R' (a) \) and \( R'' (\beta) \) are both of them true, \( R''' (\gamma) \) is true." Then such a law establishes a functional relation, or a system of functional relations, amongst the various series from which the elements of \( a, \beta, \text{and} \gamma \), respectively, are chosen.

For instance, \( R' (a) \) may stand for some combination of quanti-
ties of different forms of physical energy (coal burned, water-power supplied, etc.). These forms of energy may be combined in the production of certain industrial products. Each of these quantities, in a special case, will then be a member of its own series (weight of coal, amount of water used at a certain “head,” etc.). R'' (β) may be a combination of the costs of these various forms of energy, when the energy is obtained under certain conditions. And then, again, each of these elements of cost will have its place in its own series, determined by a price-list (price of coal per ton, of water per cubic metre, etc.). Hereupon, in ways determined by the mode of production, by the waste or the use of energy, etc., there may correspond to a given combination R' (α) and a combination R'' (β), a given set of costs of a set of industrial products, expressed by R''' (γ). In such a case the costs of the products will appear as in “functional relations” to the sources of energy used, and to the costs of each of these sources. Wherever such a correlation of series, or of sets or systems of series appears, the result is an Order determined by the correlations.

As Klein long since pointed out, the various types of Geometrical Science, the different geometries (metrical, projective, etc.), may be classified in terms of the “invariants” (that is, of the unchanging laws of the results of correlation) to which the different geometrical “transformations” are subject. And the geometrical “transformations” (projections, systematic deformations, dualities, inversions, etc.) involve correlations of sets of series such that (with the foregoing definition of the symbols used), R' (α), and R'' (β), etc., imply, as their combined result R''' (γ), in ways which the relational properties of the geometrical world enable the geometer to define. In general a mathematical “transformation” means a definition of one system of relations by means of a correspondence with other systems of relations and of related terms. Its “invariant” is a law or a relational property or construction which is exemplified by each and by all of the correlated systems.

A very highly important condition of the orderly character of the systems within which such “functional relations,” and such “transformations” are possible, is the existence of relational properties that admit of eliminations, of the type discussed in our general account of relations (§ 18, near the close). What transitivity is in the definition of a single series, the more general relational properties which permit elimination are, in the definition of the complex
geometrical and physical order-systems which admit of definite and lawfully repeated correlations and transformations. It remains to say a word as to the significance of the Symmetrical Relations in the constitution of all such order-types. If \( a = b = c = d \), etc., the set of objects between any two of which such a dyadic symmetrical transitive relation as obtains, may be called a *Level*. On a topographical map, the lines that indicate levels, the "contour-lines," run through points any two of which stand for physical points, on the surface mapped, such that they are at equal heights above some "base-level" (usually above sea-level). Isothermal lines, isobars, parallels of latitude, and countless other symbols for levels, are conspicuous features of the diagrams that are used to depict the orderly structure of real or ideal objects. Yet the members of such a level are not ordered by means of their symmetrical transitive levelling relations. They are ordered, if at all, in terms of serial relations, or in terms of the foregoing correlations of systems of series. Yet levelling processes and relations are constantly used in the definition of order-systems. The topographical map, or the "weather map" illustrates this fact. And the vast range of usefulness which the Equation has in mathematics is one of the best known features of that science. Why are relations which by themselves do not order, so useful in the definition of types of order?

The answer to this question is three-fold:

1. The symmetrical relations, and especially the symmetrical transitive relations, enable us to classify, and so form the basis for all the most exactly definable classifications of the Science of Order.

2. For this very reason, many of the most important series in the theoretical sciences are *Series of Levels*. Such, for instance, are the series of contour-lines, isobars, etc., on a map.

3. And again, for the same reason, many of the most important *laws* of an ordered world are defined in terms of levels. The "invariants" of a system of "transformations" establish, in general, such levels. That is, when two or more systems are correlated through a "transformation," the results of such correlation leave certain relations that belong to each system unchanged by the passage from one system to the other. Thus a level is established. For instance, the law of the Conservation of Energy is a law expressed by asserting that, between any two states \( A \) and \( B \) of a given "closed system" in the physical world there obtains a certain
symmetrical transitive relation, namely the relation expressed by saying that: "The total energy present in the system in the state \( A \) is equal in quantity to the total energy present in the system in the state \( B \)." In other words, the total energy remains invariant through the transformation. Thus the statement of the "invariant" law of any system of correlations or of transformations always includes some elements that can be expressed by symmetrical transitive relations. All this is the result of the same inseparable union of the concepts of Class and of Relation—a union which we have illustrated from the beginning of our sketch of Order.

It will be noted, as we now look back, that the various norms of deductive inference, in all the various cases here in question, depend upon the relational properties of the order-systems which are under consideration, and so, in the last analysis, upon the properties of single relations. Thus Formal Logic, as a "Normative science," is incidental to the application of the Theory of Order to this or that process of deductive inference.

The Logical Genesis of the Types of Order

§ 21. In our first section, the study of methodology showed us the relation of all scientific procedure to the Theory of Order. In our second section we have portrayed, in a largely empirical fashion, the Types of Order which characterize the exact sciences. Two of the concepts absolutely essential to the Theory of Order, we have already treated, indeed, so as to show why they are necessary. These are the concepts of Relation and of Class. For not only are these concepts actually used in the definition of every type of order, but as we have seen, their necessity depends upon the fact that without them no rational activity of any kind is possible. We have consequently insisted that these concepts unite in a very characteristic way—"creation" and "discovery," an element of contingency and an element of absoluteness. That a particular physical or psychical relation, such as that of father and child, should be present in the world, is as empirical a fact as the existence of colors or tones. That there should be physical objects to classify, this again is a matter of experience. And furthermore, every classification of real or of ideal objects is determined in any special instance, by a norm or principle of classification which we voluntarily choose. And in so far classifications are arbitrary, and may
be said to be "creations" or "constructions." Yet, whatever else the world contains, if it only contains a reasonable being who knows and intends his own acts, then this being is aware of a certain relation, the relation between performing and not performing any act which he considers in advance of action. And thus relations amongst acts are in such wise necessary facts, that whoever acts at all, or whoever, even in ideal, contemplates possible courses of action, must regard at least some of these relations as present in the realm of his conceived modes of action. In a similar fashion, as we have seen, every sort of action determines a kind of classification of some world, physical or ideal. In so far, therefore, as the nature of relations and of classes in general determines the existence and the meaning of types of orderly activity, these types of orderly activity, and the order-systems which express their nature, are both empirical objects, "found" (since we experience their presence in our world); and are also necessary objects, because if we try to conceive that they are not there, our very conception involves modes of action, and hence restores these necessary relations and classes to the world from which we had tried to banish them. We "construct" relational systems and classes in our ideal world. But we also "find" that at least some of these constructions are necessary.

A frequently asserted modern view, to which we have made some reference in the foregoing, namely the view called Pragmatism, asserts that all truth, including logical truth, has its basis in the fact that our hypotheses, or other assertions, prove to be successful, or show by their empirical workings that they meet the needs which they were intended to meet. From this point of view the logical hypothesis: "That there are classes, relations, and order systems," would be true merely in so far as the acts of conceiving such objects, and of treating them as real, have, under the empirical conditions under which we do our thinking, a successful result. And thus logical truth, and the logical existence and validity of classes, of relations, and of the various types of order, would stand in the same position in which all the "working hypotheses" of an empirical science stand. These order-systems would exist, and their laws would be valid, precisely in so far as such ways of actively conceiving of the world have successful workings.

But, in the foregoing, we have already indicated that, so far as the existence of classes and of relations in general is in question,
and in so far as the validity of certain logical laws is concerned, we are obliged to maintain a position which we may characterize by the term Absolute Pragmatism. This position differs from that of the pragmatists now most in vogue. There are some truths that are known to us not by virtue of the special successes which this or that hypothesis obtains in particular instances, but by virtue of the fact that there are certain modes of activity, certain laws of the rational will, which we reinstate and verify, through the very act of attempting to presuppose that these modes of activity do not exist, or that these laws are not valid. Thus, whoever says that there are no classes whatever in his world, inevitably classifies. Whoever asserts that for him there are no real relations, and that, in particular the logical relation between affirmation and denial does not exist, so that for him yes means the same as no, on the one hand himself asserts and denies, and so makes a difference between yes and no; and, on the other hand, asserts the existence of a relational sameness even in denying the difference between yes and no.

In brief, whatever actions are such, whatever types of action are such, whatever results of activity, whatever conceptual constructions are such, that the very act of getting rid of them, or of thinking them away, logically implies their presence, are known to us indeed both empirically and pragmatically (since we note their presence and learn of them through action); but they are also absolute. And any account which succeeds in telling what they are has absolute truth. Such truth is a "construction" or "creation," for activity determines its nature. It is "found," for we observe it when we act.

It consequently follows that whoever attempts to justify the existence of any of the more complicated systems of order that we have been describing in the foregoing section, has a right to seek for some absolute criterion, whereby he may distinguish what systems of order are necessary facts in the world,—that is, in the world that the logician has a right to regard as necessary,—and what, if any, amongst these forms are either capricious, and unnecessary, or else are suggested by the particular facts of experience in such wise as to remain merely contingent.

The logician's world is the world of hypotheses, and of theories, and of the ideal constructions that are used in these theories and hypotheses. Now theories and hypotheses may be merely suggested to us by physical phenomena, so that, if we had different
sensations from our present ones, or if our perceptions followed
some other routine than the observed one, we should have no need
for these hypotheses and resulting theories. In so far, the hypothe-
eses are contingent, and the theories have only conditional value.
Furthermore, some of our activities are indeed arbitrary, so that
we may, as the common expression is, "do as we like." And when
such modes of activity play their part in the choice or in the defini-
tion of our hypotheses, the logician cannot regard them as neces-
sary. But such logical facts as the difference between yes and no,
are not dependent on the contingent aspect of our sensations, but
on our rational consciousness of what we intend to do or not to
do. Such facts have not the contingency of the empirical particu-
larss of sense. And some modes of action, such as affirmation and
denial, are absolute modes.

We can indeed suspend the process of affirmation and denial,
but only by suppressing every rational consciousness about what
we ourselves purpose to do. The particular deed may be arbitrary.
But the absolute modes of activity just suggested are not arbitrary.
We cannot choose to do without them, without seeking to choose,
since choice is action, and involves, for instance, the aforesaid
difference between affirming and denying that we mean to do
thus and thus.

§ 22. Considerations of this sort show us that the Theory of
Order must undertake a task which the foregoing sketch has only
suggested. It now appears that the logician's world has in it some
necessary elements and laws upon which order-systems may be
founded. But this fact does not of itself suffice to tell us what ones
amongst the enormously complicated order-systems of mathema-
tics include contingent and arbitrary elements, and what ones are
indeed in such wise necessary that whoever knows what his own
orderly activity is, must recognize that these order-systems belong
to his logical world. Let us illustrate the issue thus brought to our
attention.

In the physical world, we meet with the difference between
greater weights and less weights. We meet with this difference
empirically, and test it by experiment. The result is that we get
tests, such as the balance, whereby we can arrange physical weights
in a series of Levels, each level consisting of observed weights any
two of which are equal, while the series of these levels is determined
by the transitive and totally non-symmetrical relation of greater
and less. The familiar operations of putting two weights in one scale-pan of a balance and finding a single weight that, put into the other scale-pan, will balance them, enables us to define for the weights an operation of summation,—a triadic relation of weights. This operation empirically conforms to the laws of the addition of quantities. Hereupon, by processes not further to be followed in this discussion, we establish an ideal and hypothetical correlation between physical weights and the number-system of arithmetic; and so the physical world, so far as weights are concerned, is conceived in orderly terms, in a way that makes many physical theories logically possible.

Now it is obvious that the existence of physical weights, and that all of the foregoing relations, so far as they are physical relations, are, from our human point of view, both empirical and contingent. We can easily conceive of a physical world without any such phenomena. For if all our knowledge of nature came to us through sight and smell alone, in the form of colours, odours, etc., and if we never saw anything that suggested to us the comparing of weights, we should of course know of no physical facts that would define for us this order-system.

On the other hand, in defining the system of the weights as in the case of any other extensive quantities, we use our empirical facts for the sake of establishing some kind of correlation between the quantities of our physical world and the facts and laws of the number-system. But what shall we say about the number-system itself? It is a system, whose first principles can be stated as hypotheses of a very general nature concerning objects that can be distinguished, numbered, etc. Is our experience of the existence of such objects altogether as contingent as our experience of the existence of weights in the physical world? One obvious answer is suggested by the fact that we can apply the system of the whole numbers to characterize our own acts. Any orderly succession of deeds in which we pass from one to the next has certain of the characters of the series of ordinal whole numbers. In any orderly activity that we begin, we have a first act followed by a second, followed by a third, and so on. It therefore may occur to our minds that our knowledge of at least the whole numbers, like our knowledge of the difference between yes and no, may be founded upon the consciousness of our own activity and some of its necessary characters. But this view, when first stated, meets with the very
obvious difficulty that, during our actual human lives, we perform only at best a very limited number of distinct acts, while the whole number-series, as the mathematician conceives it is an infinite sequence. Furthermore, nothing about the empirical nature of our activity as human beings seems to determine the number of deeds that we shall do in our short lives. But the whole numbers of the mathematician present themselves as an order-system such that every member of the series must have its next successor. No mere observation of the contingent sequence of our own empirical deeds can therefore by itself warrant the necessity that the infinite sequence of the whole numbers should have a place in the logician's world at all.

Yet this consideration is, once more, only a suggestion of a difficulty, but not a decisive proof that the whole-number series is devoid of absolute necessity. For perhaps there is indeed something about the nature of our activity, in so far as it is rational,—something which necessitates a possible next deed after any deed that has been actually accomplished. And this possibility may prove to have something absolute about it. Such considerations deserve at least a further study.

To sum up:—The order-systems of mathematics are suggested in some cases by contingent empirical phenomena. In other aspects these order-systems may prove upon analysis to be absolutely necessary facts, in the same sense in which the existence of classes and relations of some sort are necessary facts in our world. And thus may be stated the central problem of the Theory of Order. This problem is:—What are the necessary "logical entities," and what are their necessary laws? What objects must the logician's world contain? What order-systems must be conceive, not as contingent and arbitrary, but as so implied in the nature of our rational activity that the effort to remove them from our world would inevitably imply their reinstatement, just as the effort to remove relations and classes from the world would involve recognizing both classes and relations as, in some new way, present.

It is precisely in this form that the problem of the theory of order appears to be, at the present time, undergoing a most progressive series of changes, enlargements, and enrichments. The "Deduction of the Categories" is taking on decidedly new forms in recent discussion. The principles that will enable us in the future
to make an indubitable endless progress in this field at least possible, remain very briefly to be considered as our sketch closes.

§ 23. Common to all the recent logicians who have dealt seriously with the problem thus defined is the tendency to reduce all the order-systems of mathematics to a form defined, so far as possible, in terms of a few simple and necessary "logical entities," and "fundamental hypotheses" about relational properties and about the objects whose relations are in question.

In all the older attempts to characterize the mathematical systems of an orderly type; great stress was laid upon the assumption of so-called "self-evident" Axioms. The example of Euclid in his Geometry, and the Aristotelian logical theory regarding the necessity of founding all proof upon "immediate" certainties,—these were the paramount influences in determining this tendency. But the more the logician considers the so-called "self-evident" principles of the older mathematical statements, the more reason does he see to condemn self-evidence as in itself a fitting logical guide. When we call an assertion self-evident we generally do so because we have not yet sufficiently considered the complexity of the relations involved. And many propositions have been supposed to be self-evident truths that upon closer acquaintance have turned out to be decidedly inexact in their meaning, or altogether incorrect.

In two cases, in the foregoing discussion, we have had occasion to indicate for logical purposes how inadequate the older assumptions regarding the axioms of mathematics and other sciences have been. The first case was presented to us by the presupposition of induction, to the effect that the realm of the objects of possible experience has in any of its definable collections of fact a determinate constitution. In mentioning this presupposition in § 10, we stated that it is not self-evident. In § 19 this presupposition appeared in the form of the postulate: That there are Individuals. The substantial identity of the two postulates appears upon due reflection. But, as we remarked (in § 19), the postulate: That there are individuals, is too complex to be self-evident, although, upon the other hand, a study of the conception of an individual led us to the assertion, not very fully discussed in this sketch, that this postulate is indeed at once pragmatic and absolute. As we said, in our former passage, the principle in question has metaphysical aspects that cannot here be discussed.
At all events, however, we gain, and we do not lose, by regarding the postulate of individuality not as "self-evident" but as the expression of an extremely complex, but at the same time fundamental demand of the rational will,—a demand without which our activity becomes rationally meaningless.

The other case of a so-called "axiom" was mentioned in § 18, where we spoke of the principle: That things equal to the same thing are equal to each other. We gain instead of losing when this principle no longer seems self-evident, because we have come to observe that it involves a synthesis of the logically independent characters, transitivity and symmetry,—a synthesis which always needs to be justified, either by experience or by definition, or by demonstration, or finally, if that is possible, by the method which we have already applied in dealing with the concepts of class and of relation.

As a fact, therefore, most modern investigators of the Theory of Order have abandoned the view that the fundamental types of order can be defined in terms of merely "self-evident" axioms. These investigators have therefore come to be divided, largely, into two classes: (1) those who, in company with the Pragmatists, are disposed to admit a maximum of the empirical and the contingent into the theory of order; and (2) those who are disposed, like the present writer, to regard the fundamental principles of logic as sufficient to require the existence of a realm of ideal, i.e. of possible objects, which is infinitely rich, which contains systems such as the order-system of the numbers, and which conforms to laws that are in foundation the same as the laws to which one conforms when he distinguishes between yes and no, and when he defines the logical properties of classes and relations.

The writers of the first class would maintain, for instance, that whether or no such distinctions as that between yes and no have a necessary validity over and above that which belongs to physical objects, such systems as the ordinal whole numbers are simply hypothetical generalizations from experience, are empirically known to be valid so far as our process of counting extends, and are regarded in mathematics as absolute, so to speak, by courtesy. The field within which such logical empiricists very naturally find their most persuasive instances, is the field presented by geometrical theories. Geometry is a field in which purely logical considerations, and very highly contingent physical facts and relations,
have been, in the past, brought into a most extraordinary union, which only recent research has begun to disentangle. Is Geometry at bottom a physical science? Or is it rather a branch of pure logic, the discussion of an order-system or order-systems that possess a logically ideal necessity? The modern discussion of the Principles of Geometry has indeed greatly emphasized the enormous part that a purely logical Theory of Order plays in the development of geometry. But such a theory depends after all upon assumptions. Some of these assumptions, such as the famous Euclidian postulate regarding parallels, appear to some of the writers in question to have an obviously empirical foundation, as contingent as is the physical law of gravitation, and as much subject to verifications which are only approximate as that law is.

Over against these logical empiricists there are those who, however they analyze such special cases as that of geometry, agree with Mr. Bertrand Russell (in his *Principles of Mathematics*) in viewing the pure Theory of Order as dependent upon certain "logical constants." Such "logical constants" Mr. Russell assumes to be fundamental and inevitable facts of a realistically conceived world of purely logical entities, whose relation to our will or activity Mr. Russell would indeed declare to be factitious and irrelevant. Given the "logical constants," Mr. Russell regards the order-systems as creatures of definition: although, from his point of view, definition also appears to be a process by which one reports the existence, in the logician's realm, of certain beings, namely, classes, relation, series, orders, of the degrees of complexity described in our foregoing sketch. The Theory of Order for Mr. Russell is the systematic characterization of these creatures of definition. It asserts that the properties of these systems follow from their definitions. And pure mathematics consists of propositions of the type "p implies q," propositions p and q being defined, in terms of the "logical constants," and so, that, whatever entities there be (Mr. Russell's "variables") which are defined in terms of proposition p, are also such that proposition q holds of them. In the main Mr. Russell's procedure carries out with great finish ideas already developed by the school of Peano. Mr. Russell's doctrines serve, then, as examples of logical opinions which are not, in the ordinary sense, empiristic.

But the "burning questions" already mentioned as prominent in recent logical theory have shown how difficult it is to make articu-
late the theory of Mr. Russell, the somewhat similar position of Frege in Germany, and the methods of the school of Peano, without making more "pressing" than ever the question as to what classes, series, order types, and systems are to be regarded as unquestionably existing in the world which the theory of order studies, when it abstracts from physical experience and confines itself to the entities and systems of entities which can be defined solely in terms of the "logical constants." There is no doubt of the great advance made in recent times by the writers of this school in actually working out the deductive consequences of certain postulates, when these are once used for the purpose of defining a system. And every such working out is indeed a discovery of permanent importance for the theory of order. To define, for instance, what are called ideal "space-forms," upon the basis of principles more or less similar to, or more or less general than, the postulates of Euclid, is to reach actual and positive results valid for all future Theory of Order. But as the present state of the Theory of Assemblages shows, serious doubts may rise in any one case as to whether such definitions and postulates do not involve latent contradictions, which render the theories in question inadequate to tell us what order-systems are indeed the necessary ones, and what the range of those entities is whose existence can be validated by considerations as fundamental as those which we have already used in speaking of classes and relations in general.

§ 24. One method of escape from the difficulties thus suggested is a way that, in principle, was pointed out a good many years since by an English logician, Mr. A. B. Kempe. In the year 1886, in the Philosophical Transactions of the Royal Society, Mr. A. B. Kempe published a memoir on the "Theory of Mathematical Form," in which, amongst other matters, he discussed the fundamental conceptions both of Symbolic logic and of Geometry. The ideas there indicated were further developed, by Mr. Kempe, in an extended paper "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points," in the Proceedings of the London Mathematical Society for 1890. Despite the close attention that has since then been devoted to the study of the foundations of geometry, Mr. Kempe's views have remained almost unnoticed. They concern, however, certain matters which recent research enables us to regard with increasing interest.

In 1905 the present writer published, in the Transactions of the
American Mathematical Society, a paper entitled "The Relation of the Principles of Logic to the Foundations of Geometry." This paper attempts (1) to show that the principles which Mr. Kempe developed can be stated in a different, and as the author believes, in a somewhat more precise way; and (2) that the principles in question, namely the principles which are involved in any account of the nature of logical classes and their relations, are capable of a restatement in terms of which we can define an extremely general order-system. This order-system is the one which Mr. Kempe had partially defined, but which the present author's paper attempted to characterize and develop in a somewhat novel way. The thesis of that paper, taken in conjunction with Mr. Kempe's results may be restated thus:--

Both classes and propositions are objects without which the logician cannot stir a step. Their relations and laws have therefore, in the foregoing sense, an absolute validity. But, if we state these relations as laws in a definite way, and if we thereupon define a further principle regarding the existence of certain logical entities which in many respects are similar to classes and propositions,—a principle not heretofore expressly considered by logicians,—we hereupon find ourselves forced to conceive the existence of a system called, in the paper of 1905, "The System Σ." This system has an order which is determined entirely by the fundamental laws of logic, and by the one additional principle thus mentioned. The new principle in question is precisely analogous to a principle which is fundamental in geometrical theory. This is the principle that, between any two points on a line, there is an intermediate point, so that the points on a line constitute, for geometrical theory, at least a dense series. In its application to the entities of pure logic this principle appears indeed at first sight to be extraneous and arbitrary. For the principle corresponding to the geometrical principle which defines dense series of points, does not apply at all to the logical world of propositions. And, again, it does not apply with absolute generality to the objects known as classes. But it does apply to a set of objects, to which in the foregoing repeated reference has been made. This set of objects may be defined as, "certain possible modes of action that are open to any rational being who can act at all, and who can also reflect upon his own modes of possible action." Such objects as "the modes of action" have never been regarded heretofore as logical entities in the sense
in which classes and propositions have been so regarded. But in fact our modes of action are subject to the same general laws to which propositions and classes are subject. That is:—

1. To any "mode of action," such as "to sing" or "singing" (expressed in English either by the infinitive or by the present participle of the verb) there corresponds a mode of action, which is the *contradictory* of the first, for example "not to sing" or "not singing." Thus, in this realm, to every \( x \) there corresponds *one*, and essentially *only one*, \( \bar{x} \).

2. Any pair of modes of action, such for instance as "singing" and "dancing," have their "logical product," precisely as classes have a product, and their "logical sum," again, precisely as the classes possess a sum. Thus the "mode of action" expressed by the phrase: "To sing and to dance" is the logical product of the "modes of action" "to sing" and "to dance." The mode of action expressed by the phrase, "Either to sing or to dance," is the logical sum of "to sing" and "to dance." These logical operations of addition and multiplication depend upon triadic relations of modes of action, precisely analogous to the triadic relation of classes. So then, to any \( x \) and \( y \), in this realm, there correspond \( x \cdot y \) and \( x + y \).

3. Between any two modes of action a certain dyadic, transitive and not totally non-symmetrical relation may either obtain or not obtain. This relation may be expressed by the verb "implies." It has precisely the same relational properties as the relation \( \rightarrow \) of one class or proposition to another. Thus the mode of action expressed by the phrase, "To sing and to dance," *implies* the mode of action expressed by the phrase "to sing." In other words "Singing and dancing," implies "singing."

4. There is a mode of action which may be symbolized by a 0. This mode of action may be expressed in language by the phrase, "to do nothing," or "doing nothing." There is another mode of action which may be symbolized by 1. This is the mode of action expressed in language by the phrase "to do something," that is, to act positively in any way whatever which involves "not doing nothing." The modes of action 0 and 1 are contradictories each of the other.

In consequence of these considerations, *the modes of action are a set of entities that in any case conform to the same logical laws to which classes and propositions conform.* The so-called "Algebra of Logic" may be applied to them. A set of modes of action may
therefore be viewed as a system within which the principles of logical order must be regarded as applicable.

Now it would indeed be impossible to attempt to define with any exactness "the totality of all possible modes of action." Such an attempt would meet with all the difficulties which the Theory of Assemblages has recently met with in its efforts to define certain extremely inclusive classes. Thus, just as "the class of all classes" has been shown by Mr. Russell to involve fairly obvious and elementary contradictions, and just as "the greatest possible cardinal number" in the Cantorian theory of cardinal numbers, and equally "the greatest possible ordinal number" have been shown to involve logical contradictions, so (and unquestionably) the concept of the "totality of all possible modes of action" involves a contradiction. There is in fact no such totality.

On the other hand, it is perfectly possible to define a certain set, or "logical universe" of modes of action such that all the members of this set are "possible modes of action," in case there is some rational being who is capable of performing some one single possible act, and is also capable of noting, observing, recording, in some determinate way every mode of action of which he is actually capable, and which is a mode of action whose possibility is required (that is, is made logically a necessary entity) by the single mode of action in terms of which this system of modes of action is defined. Such a special system of possible modes of action may be determined, in a precise way, by naming some one mode of action, which the rational being in question is supposed to be capable of conceiving, and of noting or recording in some reflective way any mode of action once viewed as possible. The result will be that any such system will possess its own logical order-type. And some such system must be recognized as belonging to the realm of genuinely valid possibilities by any one who is himself a rational being. The order-type of this system will therefore possess a genuine validity, a "logical reality," which cannot be questioned without abandoning the very conception of rational activity itself. For the question is not whether there exists any being who actually exemplifies these modes of activity in the same way in which "singing and dancing" are exemplified in our human world. The logical question is whether the special sets of modes of action whose logical existence as a set of possible modes of action is required (in case there is any one rational being who can conceive
of any one mode of action), is a genuinely valid system, which as such has logical existence.

Nor the logical system of such modes of action illustrates a principle, which, as just admitted, does not apply to the Calculus of Propositions. Nor does this principle apply, with complete generality, to the Calculus of Classes. But what we may here call the Calculus of Modes of Action, while it makes use of all the laws of the Algebra of Logic, also permits us to make use of the principle here in question, and in fact, in case a system of modes of action, such as has just been indicated, is to be defined at all, requires us to make use of this principle. The principle in question may be dogmatically stated thus: "If there exist two distinct modes of action p and r, such that p →< r, then there always exists a mode of action q such that p →< q →< r, while p and q are distinct modes of action and q and r are equally distinct." This principle could be otherwise stated thus "for any rational being who is able to reflect upon and to record his own modes of action, if there be given any two modes of action such that one of them implies the other, there always exists at least one determinate mode of action which is implied by the first of these modes of action and which implies the second, and which is yet distinct from both of them." That this principle holds true of the modes of action which are open to any rational being to whom any one mode of action is open, can be shown by considerations for which there is here no space, but which are of the nature heretofore repeatedly defined in this paper. For the question is not whether there actually lives any body who actually does all these things. That, from the nature of the case, is impossible. The question is as to the definition of a precisely definable set of modes of action. And this principle holds for the Calculus of possible modes of action, because, as can be shown, the denial of such a principle for a rational being of the type in question, would involve self-contradiction.

Now the consideration developed by Kempe, and further elaborated in the paper of 1905, before cited, may be applied, and in fact must be applied to the order-system of such a determinate realm of modes of action. Such a realm is in fact of the form of the foregoing system 2. A comparison of Kempe's results with the newer results developed in the author's later paper would hereupon show.—

(1) That the members, elements, or "modes of action" which
constitute this logically necessary system Σ exist in sets both finite and infinite in number, and both in "dense" series, in "continuous" series, and in fact in all possible serial types.

(2) That such systems as the whole number series, the series of the rational numbers, the real numbers, etc., consequently enter into the constitution of this system. The arithmetical continuum, for instance, is a part of the system Σ.

(3) That this system also includes in its complexities all the types of order which appear to be required by the at present recognized geometrical theories, projective and metrical.

(4) That the relations amongst the logical entities in question, namely the modes of action, of which this system Σ is composed, are not only dyadic, but in many cases polyadic in the most various way. Kempe, in fact, shows with great definiteness that the triadic relations of ordinary logic, which are used in defining "sums" and "products," are really dependent upon tetradic relations into which 0 or 1, one or both may enter. In addition to these tetradic relations the logical order-system also depends for some of its most remarkable properties upon a totally symmetrical tetradic relation that, in the sense described in § 18, is transitive by pairs. These special features of the system of logical entities are here mentioned for the sake of merely hinting how enormously complex this order-system is. The matter here cannot be further discussed in its technical details. The result of these considerations is that it at present appears to be possible to define, upon the basis of purely logical relations, and upon the basis of the foregoing principles concerning rational activity, an order-system of entities inclusive not only of objects having the relation of the number system, but also of objects illustrating the geometrical types of order, and thus apparently including all the order-systems upon which, at least at present, the theoretical natural sciences depend for the success of their deductions.

And so much must here serve as a bare indication of the problems of the Theory of Order, problems which, at the present day, are rapidly undergoing reconsideration and which form an inexhaustible topic for future research. Of the fundamental philosophical importance of such problems no student of the Categories, no one who understands the significance of Kant's great undertaking, no one who takes Truth seriously, ought to be in doubt. The Theory of Order will be a fundamental science in the philosophy of the future.