Peirce on Continuity

In a letter to William James on November 25, 1902, Peirce spoke of "the completely developed system, which all hangs together and cannot receive any proper presentation in fragments," and he went on to describe synechism as "the keystone of the arch" (8.255–257). Now, synechism, according to Peirce, is just "that tendency of philosophical thought which insists upon the idea of continuity" (6.169). Thus, it would appear essential to understand what Peirce meant by "continuity"—the master key which, he claimed, would unlock the arcana of philosophy (1.163). Here, I will show how Peirce's technical definitions changed as his thinking of continuity developed.¹

Peirce did not have a single completed definition of continuity. On the contrary, from 1880 to 1911 his attempts to give continuity a precise mathematical expression show a clear development marked by several significant changes. Four main periods may be identified: (1) pre-Cantorian: until 1884; (2) Cantorian: 1884–1894; (3) Kantistic: 1895–1908; and (4) post-Cantorian: 1908–1911. Although exact dating of these transitions is difficult, the periods are approximately correct and the general characteristics of each stage are clear.

Pre-Cantorian Period

In an article written for The American Journal of Mathematics in 1881, Peirce showed that he was still subject to the not uncommon confusion between the notions of continuity and infinite divisibility or compactness. He said, for example, that, "a continuous system is one in which every quantity greater than another is also greater

¹ An earlier version of this chapter, co-authored with Paul Shields, first appeared in the Transactions of the Charles S. Peirce Society, 13 (1977), 20–34.
than some intermediate quantity greater than that other” (3.256). It is likely that this confusion persisted until around 1884, when Peirce first read Cantor’s *Grundlagen einer allgemeinen Mannigfaltigkeitslehre* in volume two of *Acta Mathematica*.

**CANTORIAN PERIOD**

The first published definition of continuity in Peirce’s Cantorian period is that written for the *Century Dictionary* in 1889. Peirce stayed rather close to Cantor’s *Grundlagen* in this article. He followed Cantor both in claiming that the notion of continuity must be defined independently of our conceptions of time and space, and in dismissing old definitions of continuity ascribed to Aristotle and Kant. He then said that Cantor’s definition by perfect concatenation is “the less unsatisfactory definition” (6.164).

Over the next four years, Peirce proposed what are essentially modifications of the definition by perfect concatenation. It would be nice to report that he improved upon Cantor’s definition. And, actually, there does appear to be some truth in Peirce’s criticism that Cantor’s definition “ingeniously wraps up its properties in two separate parcels but does not display them to our intelligence” (6.121, 1892). A series is perfect, for Cantor, when it is both closed (abgeschlossen) and condensed in itself (insichidicht). But every concatenated system can be shown to be condensed in itself. This is why Peirce later wrote, in the margin of his personal copy of the *Century Dictionary*, that

[Cantor] defines [a perfect system] such that it contains every point in the neighborhood of an infinity of points and no other. But the latter is a character of a concatenated system; hence I omit it as a character of a perfect system. (6.167)

Yet the alternative definition of continuity, by Kanticity and Aristotelicity, which Peirce proposed in 1892 (6.121–124), 1893 (4.121), and again in 1903 (6.166), did not improve the situation. Aristotelicity, in these formulations, is a rough analogue to Cantor’s property of closure, that is, the requirement that every limiting point of the system be contained within the system. But Kanticity is just compactness or indefinite divisibility. It is, Peirce said, “having
a point between any two points" (6.166). Except under special assumptions of completeness, this property is still not strong enough to provide concatenation (Cantor's *zusammenhangend*). Concatenation, in brief, ensures that there are finite gaps in the system. Infinite divisibility by itself cannot ensure this.\(^5\)

In his 1895 memoir, "Beiträge zur Begründung der transfiniten Mengenlehre," Cantor replaced concatenation with an even stronger property, the postulate of linearity, which is required in order to keep all continua similar. But Peirce did not read this memoir until much later,\(^6\) and by 1895 his criticisms of Cantor's definition had taken on a new dimension. There was no longer the basic agreement in spirit that seemed to have prevailed until 1893–1894.

**Kantistic Period**

I have called the new position that Peirce began to formulate around 1895 "Kantistic" because Peirce discovered one of its important ingredients in Kant's definition of a continuum as "that all of whose parts have parts of the same kind" (6.168, 1903). This should not be confused with the early property of Kanticity, which is merely infinite divisibility or compactness. Rather, it implies that a continuum cannot have point-like parts at all. To come to a full understanding of this new position, though, we must look at Peirce's doctrine of "postnumeral multitudes."

By the term "multitude" Peirce meant essentially what Cantor had called the "power" (*Mächtigkeit*) of a collection. Today these are called "cardinal numbers."\(^7\) Peirce's theory of multitudes preceded, by many years, his earliest attempts to clarify the notion of ordinality. Hence, the series of transfinite or "postnumeral" multitudes that Peirce developed are those obtained solely by repeated application of Cantor's theory, that \(2^m > m\). (Peirce, incidentally, seems to have discovered the general application of diagonalization independently of Cantor; see 4.204). Thus, Peirce's entire series of multitudes, using contemporary notation,\(^8\) would look like this:

\[0, 1, 2, \ldots , \aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}} \ldots \]

Peirce confused the "postnumeral" portion of this series, at times,
with Cantor's series of cardinals, \( \aleph_0, \aleph_1, \aleph_2, \ldots \), corresponding to the original number classes. This is why he sometimes called the "primipostnumeral" multitude, \( 2\aleph \), the "smallest multitude which exceeds the denumerable multitude" (4.200–213, also 4.674). More often, Peirce was simply puzzled by the continuum hypothesis. He lacked the ordinal machinery necessary to understand the difficulty. Typical is a manuscript from about 1897 in which he first claims to prove the continuum hypothesis, and then has second thoughts, the word "prove" being crossed out and replaced with the word "argue" (MS 28).

By 1896 Peirce's theory of multitudes was sophisticated enough that he could begin to describe true continuity as coming at the end of the series of postnumeral multitudes. This was what he meant when he called a continuous collection "supermultitudinous" (MS 28) and when he said that "the possibility of determining more than any given multitude of points, or in other words, the fact that there is room for any multitude at every part of the line, makes it continuous" (3.568, 1900).

There were several philosophical motivations behind this "Kantistic" approach to defining continuity. On an intuitive level, it must have been extremely disquieting for Peirce, the synechist, to discover that the putative power of the continuum was only \( 2\aleph \). If continuity can be distinguished from compactness by greatness of multitude, Peirce must have reasoned, why should not true continuity refer to the very upper limit toward which greatness of multitude can tend? But also, as early as 1892 (6.12), Peirce was concerned with the non-metrical properties of continua. One of his passing criticisms of Cantor was that Cantor's definition "turns upon metrical considerations; while the distinction between a continuous and a discontinuous series is manifestly non-metrical" (6.121). In 1893 Peirce asked himself the question: How can continua be colored if their proper parts, points, are not colored (4.126ff.)? By placing true continuity beyond the series of postnumeral multitudes, Peirce thought that he had solved this problem (because, in that case, points cannot be regarded as the actual constituents of a continuum at all [3.568]), while retaining the relation "greater than" to allow for the possibility of determining any multitude of points whatsoever on a continuum. In one sense, continuity is totally different from any collection of discrete elements;
but in another sense, the larger such a collection becomes the more it resembles a continuum. Peirce wanted both. This is the central theme of his "Kantistic" period.

**Post-Cantorian Period**

Peirce’s “Kantistic” period extended to about 1908. The post-Cantorian period developed out of several instabilities in the “Kantistic” approach. First, there was the problem of how to interpret the relation "greater than" when it is applied to multiplicities. For a time Peirce tried to maintain that Bolzano’s technique of defining order relations among multitudes is also applicable between multitudes and multiplicities (4.178). But a correspondence between a point and a possible point tends to turn into a possible correspondence that can establish only a possible order relation. And, in 1908, Peirce discarded the notion that a continuum is actually “greater than” every discrete multitude.

Second, Peirce began to question the sense in which a continuum can be thought of as a collection at all. This doubt was present as early as 1900 when Peirce suggested in a letter to the editor of *Science* that, because collections have multitude and obey Cantor’s theorem, a continuum is not really a collection (3.568). But if a continuum is not a collection, Peirce must have developed some other way of explaining how the parts of a continuum come together as a whole—for a continuum clearly does have parts. His solution is anticipated in a passage from “The Bedrock Beneath Pragmatism,” written in 1906. Peirce proposed the definition, “Whatever is continuous has *material parts,*” emphasizing that a continuum should not be thought of as a collection of points (6.174). He then explained that the *mode of connection* between these parts contributes to the nature of the whole. In a collection, this mode of connection is just “co-being” (6.174), but in a continuum it may consist of something further.

But what this further connection is was not really made explicit until the addendum, dated May 26, 1908, to the note on continuity (4.639). Peirce wrote, “In going over the proofs of this paper, written nearly a year ago, I can announce that I have, in the interval, taken a considerable stride toward the solution of the question of
continuity" (4.642). He then described a version of Kant's definition, according to which "all of the parts of a perfect continuum have the same dimensionality as the whole." This requires not only that all the parts have parts of the same kind, but that sufficiently small parts have a uniform mode of immediate connection. Such immediate connection has as its paradigm the notion of time. In a word, Peirce had come full circle since 1889, when he agreed with Cantor that the notion of continuity should be treated independently of the notion of time. His post-Cantorian definition of continuity can be stated completely in terms of the time-like mode of immediate connection that obtains between sufficiently small time-like parts. Peirce's continuum became indifferent to multitude and thoroughly non-metrical. Pierce still held this view in 1911 (3.631).

Peirce's final conception of continuity, then, comes to this: the accepted mathematical definition of "continuity" describes an "imperfect continuum" (4.642, and see 6.276, 6.168), but the "true continuum" (6.170) is something "other than" any metrical or even ordinal relation of elements. The true continuum has no actual element.

Notes

1. One of the most important treatments to date is Murray G. Murphy, *The Development of Peirce's Philosophy* (Cambridge, Mass.: Harvard University Press, 1961), pp. 260ff.

2. Another example of this same confusion apparently occurs in "The Doctrine of Chances" (2.646). The original 1878 version says that continuity is "the passage from one form to another by insensible degrees." This was amended in 1893 to the effect that continuity only suggests the idea of limitless intermediation, that is, of compactness. And another 1893 note implies that there are other ideas besides this one involved in the notion of continuity. The presence of these corrections make this passage an interesting exhibit of the contrast between periods (1) and (2).

3. This was originally the fifth of a series of papers entitled "Über unendliche lineare Punktmannigfaltigkeiten," written in 1882, and published in 1883 in Volume 21 of *Mathematische Annalen*. It was reprinted, with an added preface and the full title, *Grundlagen einer allgemeinen Mannigfaltigkeitslehre: Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen*, in Leipzig in 1883. Portions of this latter were translated into French in *Acta Mathematica*, 2 (1884). See Georg Cantor,
Contributions to the Founding of the Theory of Transfinite Numbers, trans. Philip E. B. Jourdain (New York: Dover, 1955), p. 54, note. For references to Cantor's influence, see 3.563, 4.331, 6.223, 6.175, MS 316A–S.

4. The property of being condensed in itself is such that all the points of the system must be limiting points. Given the provision that it is specified whether these are to be upper or lower limiting points, this is equivalent to the property of compactness. And concatenation clearly implies compactness. Without this provision it is still true that compactness would imply being condensed in itself, even though the reverse implication would not necessarily hold, as in the case when a decimal ending in an infinite sequence of nines is distinguished from the decimal in which those nines are replaced with zeros and the preceding place increased by one unit. So, in any case, concatenation implies being condensed in itself. A good explanation of this is to be found in Bertrand Russell, Introduction to Mathematical Philosophy (New York: Macmillan, 1919), chap. 11; and Principles of Mathematics (New York: Norton, 1903), chap. 35 ("Cantor's First Definition of Continuity"), pp. 285–95.

5. Concatenation, according to Cantor, is the property of a collection such that if \( t \) and \( t' \) are any two of its points and epsilon is a given arbitrarily small positive number, a finite number of points, \( t_1, t_2, \ldots, t_n \) of \( P \) exist such that the distances \( tt_1, t_1t_2, \ldots, t_nt' \) are all less than epsilon. While every concatenated collection is also compact, it is not the case that every compact collection is concatenated. For example, the series found by 0 and \( 2-m/n \), where \( m \) and \( n \) are integers such that \( m \) is less than \( n \), is compact, that is, infinitely divisible, but not concatenated since the steps between 0 and any other point cannot all be made less than 1. The special conditions necessary to ensure that compact series are concatenated are spelled out in Russell, Principles, pp. 289, 290.

6. Cantor described the postulate of linearity as the property according to which an aggregate M "contains an aggregate S with the cardinal number \( \alpha \), which bears such a relation to M that between any two elements \( m_0 \) and \( m_1 \) of M elements of S lie." Cantor, Contributions, p. 134. In 1900 Peirce said that he "never had an opportunity sufficiently to examine" these memoirs (3.563), but by 1911 he was quoting from the 1895 memoir (see 3.632).

7. Murphey, Development, pp. 251ff. Murphey is not sure that Peirce meant by "multitude" what Cantor meant by "cardinal number," but simply does not like Cantor's choice of words.

8. Unfortunately, the editors of the Collected Papers have substituted Cantor's aleph symbol, \( \aleph \), for Peirce's original manuscript notation, which appears in MS 25. Peirce clearly meant to indicate the denumerable multitude (see 4.204), so I have used the more common symbol, \( \alpha \).