When the going gets tough, the tough lower their standards. This idea, the theme of the whole book, underlies the final street-fighting tool of reasoning by analogy. Its advice is simple: Faced with a difficult problem, construct and solve a similar but simpler problem—an analogous problem. Practice develops fluency. The tool is introduced in spatial trigonometry (Section 6.1); sharpened on solid geometry and topology (Section 6.2); then applied to discrete mathematics (Section 6.3) and, in the farewell example, to an infinite transcendental sum (Section 6.4).

6.1 Spatial trigonometry: The bond angle in methane

The first analogy comes from spatial trigonometry. In methane (chemical formula CH$_4$), a carbon atom sits at the center of a regular tetrahedron, and one hydrogen atom sits at each vertex. What is the angle $\theta$ between two carbon–hydrogen bonds?

Angles in three dimensions are hard to visualize. Try, for example, to imagine and calculate the angle between two faces of a regular tetrahedron. Because two-dimensional angles are easy to visualize, let’s construct and analyze an analogous planar molecule. Knowing its bond angle might help us guess methane’s bond angle.
Should the analogous planar molecule have four or three hydrogens?

Four hydrogens produce four bonds which, when spaced regularly in a plane, produce two different bond angles. In contrast, methane contains only one bond angle. Therefore, using four hydrogens alters a crucial feature of the original problem. The likely solution is to construct the analogous planar molecule using only three hydrogens.

Three hydrogens arranged regularly in a plane create only one bond angle: \[ \theta = 120^\circ. \] Perhaps this angle is the bond angle in methane! One data point, however, is a thin reed on which to hang a prediction for higher dimensions. The single data point for two dimensions \((d = 2)\) is consistent with numerous conjectures—for example, that in \(d\) dimensions the bond angle is \(120^\circ\) or \((60d)^\circ\) or much else.

Selecting a reasonable conjecture requires gathering further data. Easily available data comes from an even simpler yet analogous problem: the one-dimensional, linear molecule \(\text{CH}_2\). Its two hydrogens sit opposite one another, so the two C–H bonds form an angle of \(\theta = 180^\circ\).

Based on the accumulated data, what are reasonable conjectures for the three-dimensional angle \(\theta_3\)?

The one-dimensional molecule eliminates the conjecture that \(\theta_d = (60d)^\circ\). It also suggests new conjectures—for example, that \(\theta_d = (240 - 60d)^\circ\) or \(\theta_d = 360^\circ/(d + 1)\). Testing these conjectures is an ideal task for the method of easy cases. The easy-cases test of higher dimensions (high \(d\)) refutes the conjecture that \(\theta_d = (240 - 60d)^\circ\). For high \(d\), it predicts implausible bond angles—namely, \(\theta = 0\) for \(d = 4\) and \(\theta < 0\) for \(d > 4\).

Fortunately, the second suggestion, \(\theta_d = 360^\circ/(d + 1)\), passes the same easy-cases test. Let’s continue to test it by evaluating its prediction for methane—namely, \(\theta_3 = 90^\circ\). Imagine then a big brother of methane: a \(\text{CH}_6\) molecule with carbon at the center of a cube and six hydrogens at the face centers. Its small bond angle is \(90^\circ\). (The other bond angle is \(180^\circ\).) Now remove two hydrogens to turn \(\text{CH}_6\) into \(\text{CH}_4\), evenly spreading out the remaining four hydrogens. Reducing the crowding raises the small bond angle above \(90^\circ\)—and refutes the prediction that \(\theta_3 = 90^\circ\).
Problem 6.1 How many hydrogens?
How many hydrogens are needed in the analogous four- and five-dimensional bond-angle problems? Use this information to show that \( \theta_4 > 90^\circ \). Is \( \theta_d > 90^\circ \) for all \( d \)?

The data so far have refuted the simplest rational-function conjectures \((240 - 60d)^\circ \) and \( 360^\circ / (d+1) \). Although other rational-function conjectures might survive, with only two data points the possibilities are too vast. Worse, \( \theta_d \) might not even be a rational function of \( d \).

Progress requires a new idea: The bond angle might not be the simplest variable to study. An analogous difficulty arises when conjecturing the next term in the series 3, 5, 11, 29, …

What is the next term in the series?

At first glance, the numbers seems almost random. Yet subtracting 2 from each term produces 1, 3, 9, 27, … Thus, in the original series the next term is likely to be 83. Similarly, a simple transformation of the \( \theta_d \) data might help us conjecture a pattern for \( \theta_d \).

What transformation of the \( \theta_d \) data produces simple patterns?

The desired transformation should produce simple patterns and have aesthetic or logical justification. One justification is the structure of an honest calculation of the bond angle, which can be computed as a dot product of two C–H vectors (Problem 6.3). Because dot products involve cosines, a worthwhile transformation of \( \theta_d \) is \( \cos \theta_d \).

This transformation simplifies the data: The \( \cos \theta_d \) series begins simply \(-1, -1/2, \ldots \) Two plausible continuations are \(-1/4 \) or \(-1/3 \); they correspond, respectively, to the general term \(-1/2^{d-1} \) or \(-1/d \).

Which continuation and conjecture is the more plausible?

Both conjectures predict \( \cos \theta < 0 \) and therefore \( \theta_d > 90^\circ \) (for all \( d \)). This shared prediction is encouraging (Problem 6.1); however, being shared means that it does not distinguish between the conjectures.

Does either conjecture match the molecular geometry?

An important geometric feature, apart from the bond angle, is the position of the carbon. In one dimension, it lies halfway
between the two hydrogens, so it splits the H–H line segment into two pieces having a 1:1 length ratio.

In two dimensions, the carbon lies on the altitude that connects one hydrogen to the midpoint of the other two hydrogens. The carbon splits the altitude into two pieces having a 1:2 length ratio.

How does the carbon split the analogous altitude of methane?

In methane, the analogous altitude runs from the top vertex to the center of the base. The carbon lies at the mean position and therefore at the mean height of the four hydrogens. Because the three base hydrogens have zero height, the mean height of the four hydrogens is \( h/4 \), where \( h \) is the height of the top hydrogen. Thus, in three dimensions, the carbon splits the altitude into two parts having a length ratio of \( h/4 : 3h/4 \) or 1:3. In \( d \) dimensions, therefore, the carbon probably splits the altitude into two parts having a length ratio of 1:d (Problem 6.2).

Because 1:d arises naturally in the geometry, \( \cos \theta_d \) is more likely to contain \( 1/d \) rather than \( 1/2^{d-1} \). Thus, the more likely of the two \( \cos \theta_d \) conjectures is that

\[
\cos \theta_d = -\frac{1}{d}.
\] (6.1)

For methane, where \( d = 3 \), the predicted bond angle is \( \arccos(-1/3) \) or approximately 109.47°. This prediction using reasoning by analogy agrees with experiment and with an honest calculation using analytic geometry (Problem 6.3).

**Problem 6.2  Carbon’s position in higher dimensions**

Justify conjecture that the carbon splits the altitude into two pieces having a length ratio 1:d.

**Problem 6.3  Analytic-geometry solution**

In order to check the solution using analogy, use analytic geometry as follows to find the bond angle. First, assign coordinates \((x_n, y_n, z_n)\) to the \( n \) hydrogens, where \( n = 1 \ldots 4 \), and solve for those coordinates. (Use symmetry to make the coordinates as simple as you can.) Then choose two C–H vectors and compute the angle that they subtend.
Problem 6.4 Extreme case of high dimensionality

Draw a picture to explain the small-angle approximation $\arccos x \approx \pi/2 - x$. What is the approximate bond angle in high dimensions (large $d$)? Can you find an intuitive explanation for the approximate bond angle?

6.2 Topology: How many regions?

The bond angle in methane (Section 6.1) can be calculated directly with analytic geometry (Problem 6.3), so reasoning by analogy does not show its full power. Therefore, try the following problem.

> Into how many regions do five planes divide space?

This formulation permits degenerate arrangements such as five parallel planes, four planes meeting at a point, or three planes meeting at a line. To eliminate these and other degeneracies, let’s place and orient the planes randomly, thereby maximizing the number of regions. The problem is then to find the maximum number of regions produced by five planes.

Five planes are hard to imagine, but the method of easy cases—using fewer planes—might produce a pattern that generalizes to five planes. The easiest case is zero planes: Space remains whole so $R(0) = 1$ (where $R(n)$ denotes the number of regions produced by $n$ planes). The first plane divides space into two halves, giving $R(1) = 2$. To add the second plane, imagine slicing an orange twice to produce four wedges: $R(2) = 4$.

What pattern(s) appear in the data?

A reasonable conjecture is that $R(n) = 2^n$. To test it, try the case $n = 3$ by slicing the orange a third time and cutting each of the four pieces into two smaller pieces; thus, $R(3)$ is indeed 8. Perhaps the pattern continues with $R(4) = 16$ and $R(5) = 32$. In the following table for $R(n)$, these two extrapolations are marked in gray to distinguish them from the verified entries.

<table>
<thead>
<tr>
<th>n</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
</tr>
</tbody>
</table>
How can the $R(n) = 2^n$ conjecture be tested further?

A direct test by counting regions is difficult because the regions are hard to visualize in three dimensions. An analogous two-dimensional problem would be easier to solve, and its solution may help test the three-dimensional conjecture. A two-dimensional space is partitioned by lines, so the analogous question is the following:

What is the maximum number of regions into which $n$ lines divide the plane?

The method of easy cases might suggest a pattern. If the pattern is $2^n$, then the $R(n) = 2^n$ conjecture is likely to apply in three dimensions.

What happens in a few easy cases?

Zero lines leave the plane whole, giving $R(0) = 1$. The next three cases are as follows (although see Problem 6.5):

- $R(1) = 2$
- $R(2) = 4$
- $R(3) = 7$

Problem 6.5  Three lines again

The $R(3) = 7$ illustration showed three lines producing seven regions. Here is another example with three lines, also in a random arrangement, but it seems to produce only six regions. Where, if anywhere, is the seventh region? Or is $R(3) = 6$?

Problem 6.6  Convexity

Must all the regions created by the lines be convex? (A region is convex if and only if a line segment connecting any two points inside the region lies entirely inside the region.) What about the three-dimensional regions created by placing planes in space?

Until $R(3)$ turned out to be 7, the conjecture $R(n) = 2^n$ looked sound. However, before discarding such a simple conjecture, draw a fourth line and carefully count the regions. Four lines make only 11 regions rather than the predicted 16, so the $2^n$ conjecture is dead.

A new conjecture might arise from seeing the two-dimensional data $R_2(n)$ alongside the three-dimensional data $R_3(n)$. 
6.2 Topology: How many regions?


\begin{align*}
  n & 0 1 2 3 4 \\
  R_2 & 1 2 4 7 11 \\
  R_3 & 1 2 4 8 \\
\end{align*}

In this table, several entries combine to make nearby entries. For example, \( R_2(1) \) and \( R_3(1) \)—the two entries in the \( n = 1 \) column—sum to \( R_2(2) \) or \( R_3(2) \). These two entries in turn sum to the \( R_3(3) \) entry. But the table has many small numbers with many ways to combine them; discarding the coincidences requires gathering further data—and the simplest data source is the analogous one-dimensional problem.

**What is the maximum number of segments into which \( n \) points divide a line?**

A tempting answer is that \( n \) points make \( n \) segments. However, an easy case—that one point produces two segments—reduces the temptation. Rather, \( n \) points make \( n + 1 \) segments. That result generates the \( R_1 \) row in the following table.

\begin{align*}
  n & 0 1 2 3 4 5 \quad n \\
  R_1 & 1 2 3 4 5 6 \quad n + 1 \\
  R_2 & 1 2 4 7 11 \\
  R_3 & 1 2 4 8 \\
\end{align*}

**What patterns are in these data?**

The \( 2^n \) conjecture survives partially. In the \( R_1 \) row, it fails starting at \( n = 2 \). In the \( R_2 \) row, it fails starting at \( n = 3 \). Thus in the \( R_3 \) row, it probably fails starting at \( n = 4 \), making the conjectures \( R_3(4) = 16 \) and \( R_3(5) = 32 \) improbable. My personal estimate is that, before seeing these failures, the probability of the \( R_3(4) = 16 \) conjecture was 0.5; but now it falls to at most 0.01. (For more on estimating and updating the probabilities of conjectures, see the important works on plausible reasoning by Corfield [11], Jaynes [21], and Polya [36].)

In better news, the apparent coincidences contain a robust pattern:

\begin{align*}
  n & 0 1 2 3 4 5 \quad n \\
  R_1 & 1 2 3 4 5 6 \quad n + 1 \\
  R_2 & 1 2 4 7 11 \\
  R_3 & 1 2 4 8 \\
\end{align*}
If the pattern continues, into how many regions can five planes divide space?

According to the pattern,

$$R_3(4) = R_2(3) + R_3(3) = 15$$  \hspace{1cm} (6.2)

and then

$$R_3(5) = R_2(4) + R_3(4) = 26.$$  \hspace{1cm} (6.3)

Thus, five planes can divide space into a maximum of 26 regions.

This number is hard to deduce by drawing five planes and counting the regions. Furthermore, that brute-force approach would give the value of only $R_3(5)$, whereas easy cases and analogy give a method to compute any entry in the table. They thereby provide enough data to conjecture expressions for $R_2(n)$ (Problem 6.9), for $R_3(n)$ (Problem 6.10), and for the general entry $R_d(n)$ (Problem 6.12).

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**Problem 6.7 Checking the pattern in two dimensions**

The conjectured pattern predicts $R_2(5) = 16$: that five lines can divide the plane into 16 regions. Check the conjecture by drawing five lines and counting the regions.

**Problem 6.8 Free data from zero dimensions**

Because the one-dimensional problem gave useful data, try the zero-dimensional problem. Extend the pattern for the $R_3$, $R_2$, and $R_1$ rows upward to construct an $R_0$ row. It gives the number of zero-dimensional regions (points) produced by partitioning a point with $n$ objects (of dimension $-1$). What is $R_0$ if the row is to follow the observed pattern? Is that result consistent with the geometric meaning of trying to subdivide a point?

**Problem 6.9 General result in two dimensions**

The $R_0$ data fits $R_0(n) = 1$ (Problem 6.8), which is a zeroth-degree polynomial. The $R_1$ data fits $R_1(n) = n + 1$, which is a first-degree polynomial. Therefore, the $R_2$ data probably fits a quadratic.

Test this conjecture by fitting the data for $n = 0 \ldots 2$ to the general quadratic $An^2 + Bn + C$, repeatedly taking out the big part (Chapter 5) as follows.

a. Guess a reasonable value for the quadratic coefficient $A$. Then take out (subtract) the big part $An^2$ and tabulate the leftover, $R_2(n) − An^2$, for $n = 0 \ldots 2$. 

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If the leftover is not linear in \( n \), then a quadratic term remains or too much was removed. In either case, adjust \( A \).

b. Once the quadratic coefficient \( A \) is correct, use an analogous procedure to find the linear coefficient \( B \).

c. Similarly solve for the constant coefficient \( C \).

d. Check your quadratic fit against new data (\( R_2(n) \) for \( n \geq 3 \)).

**Problem 6.10  General result in three dimensions**

A reasonable conjecture is that the \( R_3 \) row matches a cubic (Problem 6.9). Use taking out the big part to fit a cubic to the \( n = 0 \ldots 3 \) data. Does it produce the conjectured values \( R_3(4) = 15 \) and \( R_3(5) = 26 \)?

**Problem 6.11  Geometric explanation**

Find a geometric explanation for the observed pattern. *Hint:* Explain first why the pattern generates the \( R_2 \) row from the \( R_1 \) row; then generalize the reason to explain the \( R_3 \) row.

**Problem 6.12  General solution in arbitrary dimension**

The pattern connecting neighboring entries of the \( \mathbf{R}_d(n) \) table is the pattern that generates Pascal’s triangle [17]. Because Pascal’s triangle produces binomial coefficients, the general expression \( \mathbf{R}_d(n) \) should contain binomial coefficients.

Therefore, use binomial coefficients to express \( \mathbf{R}_0(n) \) (Problem 6.8), \( \mathbf{R}_1(n) \), and \( \mathbf{R}_2(n) \) (Problem 6.9). Then conjecture a binomial-coefficient form for \( \mathbf{R}_3(n) \) and \( \mathbf{R}_d(n) \), checking the result against Problem 6.10.

**Problem 6.13  Power-of-2 conjecture**

Our first conjecture for the number of regions was \( \mathbf{R}_d(n) = 2^n \). In three dimensions, it worked until \( n = 4 \). In \( d \) dimensions, show that \( \mathbf{R}_d(n) = 2^n \) for \( n \leq d \) (perhaps using the results of Problem 6.12).

### 6.3 Operators: Euler–MacLaurin summation

The next analogy studies unusual functions. Most functions turn numbers into other numbers, but special kinds of functions—operators—turn functions into other functions. A familiar example is the derivative operator \( D \). It turns the sine function into the cosine function, or the hyperbolic sine function into the hyperbolic cosine function. In operator notation, \( D(\sin) = \cos \) and \( D(\sinh) = \cosh \); omitting the parentheses gives the less cluttered expression \( D \sin = \cos \) and \( D \sinh = \cosh \). To understand and learn how to use operators, a fruitful tool is reasoning by analogy: Operators behave much like ordinary functions or even like numbers.
6.3.1 Left shift

Like a number, the derivative operator \( D \) can be squared to make \( D^2 \) (the second-derivative operator) or to make any integer power of \( D \). Similarly, the derivative operator can be fed to a polynomial. In that usage, an ordinary polynomial such as \( P(x) = x^2 + x/10 + 1 \) produces the operator polynomial \( P(D) = D^2 + D/10 + 1 \) (the differential operator for a lightly damped spring–mass system).

How far does the analogy to numbers extend? For example, do \( \cosh D \) or \( \sin D \) have a meaning? Because these functions can be written using the exponential function, let’s investigate the operator exponential \( e^D \).

▶ What does \( e^D \) mean?

The direct interpretation of \( e^D \) is that it turns a function \( f \) into \( e^{Df} \).

\[
\begin{array}{c}
\text{f} \\
\xrightarrow{D} \\
\xrightarrow{Df} \\
\xrightarrow{\exp} \\
\xrightarrow{e^{Df}}
\end{array}
\]

However, this interpretation is needlessly nonlinear. It turns \( 2f \) into \( e^{2Df} \), which is the square of \( e^{Df} \), whereas a linear operator that produces \( e^{Df} \) from \( f \) would produce \( 2e^{Df} \) from \( 2f \). To get a linear interpretation, use a Taylor series—as if \( D \) were a number—to build \( e^D \) out of linear operators.

\[
e^D = 1 + D + \frac{1}{2} D^2 + \frac{1}{6} D^3 + \cdots \quad (6.4)
\]

▶ What does this \( e^D \) do to simple functions?

The simplest nonzero function is the constant function \( f = 1 \). Here is that function being fed to \( e^D \):

\[
\underbrace{(1 + D + \cdots)}_{e^D} \underbrace{1}_{f} = 1. \quad (6.5)
\]

The next simplest function \( x \) turns into \( x + 1 \).

\[
\left(1 + \frac{D^2}{2} + \cdots\right) x = x + 1. \quad (6.6)
\]

More interestingly, \( x^2 \) turns into \( (x + 1)^2 \).

\[
\left(1 + \frac{D^2}{2} + \frac{D^3}{6} \cdots\right) x^2 = x^2 + 2x + 1 = (x + 1)^2. \quad (6.7)
\]
Problem 6.14  Continue the pattern
What is $e^D x^3$ and, in general, $e^D x^n$?

What does $e^D$ do in general?

The preceding examples follow the pattern $e^D x^n = (x+1)^n$. Because most functions of $x$ can be expanded in powers of $x$, and $e^D$ turns each $x^n$ term into $(x+1)^n$, the conclusion is that $e^D$ turns $f(x)$ into $f(x+1)$. Amazingly, $e^D$ is simply $L$, the left-shift operator.

Problem 6.15  Right or left shift
Draw a graph to show that $f(x) \rightarrow f(x+1)$ is a left rather than a right shift. Apply $e^{-D}$ to a few simple functions to characterize its behavior.

Problem 6.16  Operating on a harder function
Apply the Taylor expansion for $e^D$ to $\sin x$ to show that $e^D \sin x = \sin (x+1)$.

Problem 6.17  General shift operator
If $x$ has dimensions, then the derivative operator $D = d/dx$ is not dimensionless, and $e^D$ is an illegal expression. To make the general expression $e^{aD}$ legal, what must the dimensions of $a$ be? What does $e^{aD}$ do?

6.3.2 Summation

Just as the derivative operator can represent the left-shift operator (as $L = e^D$), the left-shift operator can represent the operation of summation. This operator representation will lead to a powerful method for approximating sums with no closed form.

Summation is analogous to the more familiar operation of integration. Integration occurs in definite and indefinite flavors: Definite integration is equivalent to indefinite integration followed by evaluation at the limits of integration. As an example, here is the definite integration of $f(x) = 2x$.

\[ \int_{a}^{b} 2x \, dx = x^2 + C \]

In general, the connection between an input function $g$ and the result of indefinite integration is $DG = g$, where $D$ is the derivative operator and $G = \int g$ is the result of indefinite integration. Thus $D$ and $\int$ are inverses.
of one another—$D\int = 1$ or $D = 1/\int$—a connection represented by the loop in the diagram. ($\int D \neq 1$ because of a possible integration constant.)

![Diagram of integration and summation analogy]

**What is the analogous picture for summation?**

Analogously to integration, define definite summation as indefinite summation and then evaluation at the limits. But apply the analogy with care to avoid an off-by-one or fencepost error (Problem 2.24). The sum $\sum_2^4 f(k)$ includes three rectangles—$f(2)$, $f(3)$, and $f(4)$—whereas the definite integral $\int_2^4 f(k) \,dk$ does not include any of the $f(4)$ rectangle. Rather than rectifying the discrepancy by redefining the familiar operation of integration, interpret indefinite summation to exclude the last rectangle. Then indefinite summation followed by evaluating at the limits $a$ and $b$ produces a sum whose index ranges from $a$ to $b - 1$.

As an example, take $f(k) = k$. Then the indefinite sum $\sum f$ is the function $F$ defined by $F(k) = k(k - 1)/2 + C$ (where $C$ is the constant of summation). Evaluating $F$ between 0 and $n$ gives $n(n - 1)/2$, which is $\sum_0^{n-1} k$. In the following diagram, these steps are the forward path.

![Diagram of summation steps]

In the reverse path, the new $\Delta$ operator inverts $\Sigma$ just as differentiation inverts integration. Therefore, an operator representation for $\Delta$ provides one for $\Sigma$. Because $\Delta$ and the derivative operator $D$ are analogous, their representations are probably analogous. A derivative is the limit

$$\frac{df}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}. \quad (6.8)$$
The derivative operator $D$ is therefore the operator limit
\[ D = \lim_{h \to 0} \frac{L_h - 1}{h}, \tag{6.9} \]
where the $L_h$ operator turns $f(x)$ into $f(x + h)$—that is, $L_h$ left shifts by $h$.

**Problem 6.18 Operator limit**
Explain why $L_h \approx 1 + h D$ for small $h$. Show therefore that $L = e^{D}$.

What is an analogous representation of $\Delta$?

The operator limit for $D$ uses an infinitesimal left shift; correspondingly, the inverse operation of integration sums rectangles of infinitesimal width. Because summation $\Sigma$ sums rectangles of unit width, its inverse $\Delta$ should use a unit left shift—namely, $L_h$ with $h = 1$. As a reasonable conjecture,
\[ \Delta = \lim_{h \to 1} \frac{L_h - 1}{h} = L - 1. \tag{6.10} \]
This $\Delta$—called the finite-difference operator—is constructed to be $1/\Sigma$. If the construction is correct, then $(L - 1)\Sigma$ is the identity operator 1. In other words, $(L - 1)\Sigma$ should turn functions into themselves.

How well does this conjecture work in various easy cases?

To test the conjecture, apply the operator $(L - 1)\Sigma$ first to the easy function $g = 1$. Then $\Sigma g$ is a function waiting to be fed an argument, and $(\Sigma g)(k)$ is the result of feeding it $k$. With that notation, $(\Sigma g)(k) = k + C$. Feeding this function to the $L - 1$ operator reproduces $g$.
\[ \left[(L - 1)\Sigma g\right](k) = \frac{(k + 1 + C) - (k + C)}{(\Sigma g)(k)} = \frac{1}{g(k)}. \tag{6.11} \]

With the next-easiest function—defined by $g(k) = k$—the indefinite sum $(\Sigma g)(k)$ is $k(k - 1)/2 + C$. Passing $\Sigma g$ through $L - 1$ again reproduces $g$.
\[ \left[(L - 1)\Sigma g\right](k) = \frac{(k + 1)k/2 + C}{(\Sigma g)(k)} - \frac{k(k - 1)/2 + C}{(\Sigma g)(k)} = \frac{k}{g(k)}. \tag{6.12} \]
In summary, for the test functions $g(k) = 1$ and $g(k) = k$, the operator product $(L - 1)\Sigma$ takes $g$ back to itself, so it acts like the identity operator.
This behavior is general—\((L−1)\Sigma 1\) is indeed 1, and \(\Sigma = 1/(L−1)\). Because \(L = e^D\), we have \(\Sigma = 1/(e^D−1)\). Expanding the right side in a Taylor series gives an amazing representation of the summation operator.

\[
\sum = \frac{1}{e^D−1} = \frac{1}{D} − \frac{1}{2} + \frac{D}{12} − \frac{D^3}{720} + \frac{D^5}{30240} − \cdots. \tag{6.13}
\]

Because \(D∫ = 1\), the leading term \(1/D\) is integration. Thus, summation is approximately integration—a plausible conclusion indicating that the operator representation is not nonsense.

Applying this operator series to a function \(f\) and then evaluating at the limits \(a\) and \(b\) produces the Euler–MacLaurin summation formula

\[
\sum_a^b f(k) = \int_a^b f(k) \, dk - \frac{f(b) - f(a)}{2} + \frac{f^{(1)}(b) - f^{(1)}(a)}{12} - \frac{f^{(3)}(b) - f^{(3)}(a)}{720} + \frac{f^{(5)}(b) - f^{(5)}(a)}{30240} - \cdots, \tag{6.14}
\]

where \(f^{(n)}\) indicates the \(n\)th derivative of \(f\).

The sum lacks the usual final term \(f(b)\). Including this term gives the useful alternative

\[
\sum_a^b f(k) = \int_a^b f(k) \, dk + \frac{f(b) + f(a)}{2} + \frac{f^{(1)}(b) - f^{(1)}(a)}{12} - \frac{f^{(3)}(b) - f^{(3)}(a)}{720} + \frac{f^{(5)}(b) - f^{(5)}(a)}{30240} - \cdots. \tag{6.15}
\]

As a check, try an easy case: \(\sum_0^n k\). Using Euler–MacLaurin summation, \(f(k) = k\), \(a = 0\), and \(b = n\). The integral term then contributes \(n^2/2\); the constant term \([f(b) + f(a)]/2\) contributes \(n/2\); and later terms vanish. The result is familiar and correct:

\[
\sum_0^n k = \frac{n^2}{2} + \frac{n}{2} + 0 = \frac{n(n + 1)}{2}. \tag{6.16}
\]

A more stringent test of Euler–MacLaurin summation is to approximate \(\ln n!\), which is the sum \(\sum_1^n \ln k\) (Section 4.5). Therefore, sum \(f(k) = \ln k\) between the (inclusive) limits \(a = 1\) and \(b = n\). The result is

\[
\sum_1^n \ln k = \int_1^n \ln k \, dk + \frac{\ln n}{2} + \cdots. \tag{6.17}
\]
6.4 Tangent roots: A daunting transcendental sum

The integral, from the $1/D$ operator, contributes the area under the $\ln k$ curve. The correction, from the $1/2$ operator, incorporates the triangular protrusions (Problem 6.20). The ellipsis includes the higher-order corrections (Problem 6.21)—hard to evaluate using pictures (Problem 4.32) but simple using Euler–MacLaurin summation (Problem 6.21).

Problem 6.19 Integer sums
Use Euler–MacLaurin summation to find closed forms for the following sums:

(a) $\sum_{0}^{n} k^2$ (b) $\sum_{0}^{n} (2k + 1)$ (c) $\sum_{0}^{n} k^3$.

Problem 6.20 Boundary cases
In Euler–MacLaurin summation, the constant term is $[f(b) + f(a)]/2$—one-half of the first term plus one-half of the last term. The picture for summing $\ln k$ (Section 4.5) showed that the protrusions are approximately one-half of the last term, namely $\ln n$. What, pictorially, happened to one-half of the first term?

Problem 6.21 Higher-order terms
Approximate $\ln 5!$ using Euler–MacLaurin summation.

Problem 6.22 Basel sum
The Basel sum $\sum_{1}^{\infty} n^{-2}$ may be approximated with pictures (Problem 4.37).

However, the approximation is too crude to help guess the closed form. As Euler did, use Euler–MacLaurin summation to improve the accuracy until you can confidently guess the closed form. Hint: Sum the first few terms explicitly.

6.4 Tangent roots: A daunting transcendental sum

Our farewell example, chosen because its analysis combines diverse street-fighting tools, is a difficult infinite sum.

Find $S \equiv \sum x_n^{-2}$ where the $x_n$ are the positive solutions of $\tan x = x$.

The solutions to $\tan x = x$ or, equivalently, the roots of $\tan x - x$, are transcendental and have no closed form, yet a closed form is required for almost every summation method. Street-fighting methods will come to our rescue.
6.4.1 Pictures and easy cases

Begin the analysis with a hopefully easy case.

What is the first root \( x_1 \)?

The roots of \( \tan x - x \) are given by the intersections of \( y = x \) and \( y = \tan x \). Surprisingly, no intersection occurs in the branch of \( \tan x \) where \( 0 < x < \pi/2 \) (Problem 6.23); the first intersection is just before the asymptote at \( x = 3\pi/2 \). Thus, \( x_1 \approx 3\pi/2 \).

**Problem 6.23  No intersection with the main branch**

Show symbolically that \( \tan x = x \) has no solution for \( 0 < x < \pi/2 \). (The result looks plausible pictorially but is worth checking in order to draw the picture.)

Where, approximately, are the subsequent intersections?

As \( x \) grows, the \( y = x \) line intersects the \( y = \tan x \) graph ever higher and therefore ever closer to the vertical asymptotes. Therefore, make the following asymptote approximation for the big part of \( x_n \):

\[
    x_n \approx \left( n + \frac{1}{2} \right) \pi. \tag{6.18}
\]

6.4.2 Taking out the big part

This approximate, low-entropy expression for \( x_n \) gives the big part of \( S \) (the zeroth approximation).

\[
    S \approx \sum_{n=1}^{\infty} \left[ \left( n + \frac{1}{2} \right) \pi \right]^{-2} = \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}. \tag{6.19}
\]

The sum \( \sum_{1}^{\infty} (2n + 1)^{-2} \) is, from a picture (Section 4.5) or from Euler–MacLaurin summation (Section 6.3.2), roughly the following integral.

\[
    \sum_{1}^{\infty} (2n + 1)^{-2} \approx \int_{1}^{\infty} (2n + 1)^{-2} \, dn = -\frac{1}{2} \times \frac{1}{2n + 1} \bigg|_{1}^{\infty} = \frac{1}{6}. \tag{6.20}
\]
Therefore,

\[ S \approx \frac{4}{\pi^2} \times \frac{1}{6} = 0.067547 \ldots \]  \hspace{1cm} (6.21)

The shaded protrusions are roughly triangles, and they sum to one-half of the first rectangle. That rectangle has area \( 1/9 \), so

\[ \sum_{1}^{\infty} (2n+1)^{-2} \approx \frac{1}{6} + \frac{1}{2} \times \frac{1}{9} = \frac{2}{9}. \]  \hspace{1cm} (6.22)

Therefore, a more accurate estimate of \( S \) is

\[ S \approx \frac{4}{\pi^2} \times \frac{2}{9} = 0.090063 \ldots, \]  \hspace{1cm} (6.23)

which is slightly higher than the first estimate.

> **Is the new approximation an overestimate or an underestimate?**

The new approximation is based on two underestimates. First, the asymptote approximation \( x_n \approx (n + 0.5)\pi \) overestimates each \( x_n \) and therefore underestimates the squared reciprocals in the sum \( \sum x_n^{-2} \). Second, after making the asymptote approximation, the pictorial approximation to the sum \( \sum_{1}^{\infty} (2n+1)^{-2} \) replaces each protrusion with an inscribed triangle and thereby underestimates each protrusion (Problem 6.24).

**Problem 6.24 Picture for the second underestimate**

Draw a picture of the underestimate in the pictorial approximation

\[ \sum_{1}^{\infty} \frac{1}{(2n+1)^2} \approx \frac{1}{6} + \frac{1}{2} \times \frac{1}{9}. \]  \hspace{1cm} (6.24)

> **How can these two underestimates be remedied?**

The second underestimate (the protrusions) is eliminated by summing \( \sum_{1}^{\infty} (2n+1)^{-2} \) exactly. The sum is unfamiliar partly because its first term is the fraction \( 1/9 \)—whose arbitrariness increases the entropy of the sum. Including the \( n = 0 \) term, which is 1, and the even squared reciprocals \( 1/(2n)^2 \) produces a compact and familiar lower-entropy sum.

\[ \sum_{1}^{\infty} \frac{1}{(2n+1)^2} + 1 + \sum_{1}^{\infty} \frac{1}{(2n)^2} = \sum_{1}^{\infty} \frac{1}{n^2}. \]  \hspace{1cm} (6.25)
The final, low-entropy sum is the famous Basel sum (high-entropy results are not often famous). Its value is $B = \pi^2/6$ (Problem 6.22).

How does knowing $B = \pi^2/6$ help evaluate the original sum $\sum_1^\infty (2n + 1)^{-2}$?

The major modification from the original sum was to include the even squared reciprocals. Their sum is $B/4$.

$$\sum_1^\infty \frac{1}{(2n)^2} = \frac{1}{4} \sum_1^\infty \frac{1}{n^2}.$$ (6.26)

The second modification was to include the $n = 0$ term. Thus, to obtain $\sum_1^\infty (2n + 1)^{-2}$, adjust the Basel value $B$ by subtracting $B/4$ and then the $n = 0$ term. The result, after substituting $B = \pi^2/6$, is

$$\sum_1^\infty \frac{1}{(2n + 1)^2} = B - \frac{1}{4}B - 1 = \frac{\pi^2}{8} - 1.$$ (6.27)

This exact sum, based on the asymptote approximation for $x_n$, produces the following estimate of $S$.

$$S \approx \frac{4}{\pi^2} \sum_1^\infty \frac{1}{(2n + 1)^2} = \frac{4}{\pi^2} \left( \frac{\pi^2}{8} - 1 \right).$$ (6.28)

Simplifying by expanding the product gives

$$S \approx \frac{1}{2} - \frac{4}{\pi^2} = 0.094715 \ldots$$ (6.29)

**Problem 6.25  Check the earlier reasoning**

Check the earlier pictorial reasoning (Problem 6.24) that $1/6 + 1/18 = 2/9$ underestimates $\sum_1^\infty (2n + 1)^{-2}$. How accurate was that estimate?

This estimate of $S$ is the third that uses the asymptote approximation $x_n \approx (n + 0.5)\pi$. Assembled together, the estimates are

$$S \approx \begin{cases} 0.067547 & \text{(integral approximation to } \sum_1^\infty (2n + 1)^{-2}), \\ 0.090063 & \text{(integral approximation and triangular overshoots)}, \\ 0.094715 & \text{(exact sum of } \sum_1^\infty (2n + 1)^{-2}). \end{cases}$$

Because the third estimate incorporated the exact value of $\sum_1^\infty (2n + 1)^{-2}$, any remaining error in the estimate of $S$ must belong to the asymptote approximation itself.
For which term of $\sum x_n^{-2}$ is the asymptote approximation most inaccurate?

As $x$ grows, the graphs of $x$ and $\tan x$ intersect ever closer to the vertical asymptote. Thus, the asymptote approximation makes its largest absolute error when $n = 1$. Because $x_1$ is the smallest root, the fractional error in $x_n$ is, relative to the absolute error in $x_n$, even more concentrated at $n = 1$. The fractional error in $x_n^{-2}$, being $-2$ times the fractional error in $x_n$ (Section 5.3), is equally concentrated at $n = 1$. Because $x_1^{-2}$ is the largest at $n = 1$, the absolute error in $x_n^{-2}$ (the fractional error times $x_n^{-2}$ itself) is, by far, the largest at $n = 1$.

**Problem 6.26 Absolute error in the early terms**

Estimate, as a function of $n$, the absolute error in $x_n^{-2}$ that is produced by the asymptote approximation.

With the error so concentrated at $n = 1$, the greatest improvement in the estimate of $S$ comes from replacing the approximation $x_1 = (n + 0.5)\pi$ with a more accurate value. A simple numerical approach is successive approximation using the Newton–Raphson method (Problem 4.38). To find a root with this method, make a starting guess $x$ and repeatedly improve it using the replacement

$$x \rightarrow x - \frac{\tan x - x}{\sec^2 x - 1}. \quad (6.30)$$

When the starting guess for $x$ is slightly below the first asymptote at $1.5\pi$, the procedure rapidly converges to $x_1 = 4.4934…$

Therefore, to improve the estimate $S \approx 0.094715$, which was based on the asymptote approximation, subtract its approximate first term (its big part) and add the corrected first term.

$$S \approx S_{\text{old}} - \frac{1}{(1.5\pi)^2} + \frac{1}{4.4934^2} \approx 0.09921. \quad (6.31)$$

Using the Newton–Raphson method to refine, in addition, the $1/x_1^2$ term gives $S \approx 0.09978$ (Problem 6.27). Therefore, a highly educated guess is

$$S = \frac{1}{10}. \quad (6.32)$$

The infinite sum of unknown transcendental numbers seems to be neither transcendental nor irrational! This simple and surprising rational number deserves a simple explanation.
Problem 6.27  Continuing the corrections
Choose a small $N$, say $4$. Then use the Newton–Raphson method to compute accurate values of $x_n$ for $n = 1 \ldots N$; and use those values to refine the estimate of $S$. As you extend the computation to larger values of $N$, do the refined estimates of $S$ approach our educated guess of $1/10$?

6.4.3 Analogy with polynomials

If only the equation $\tan x - x = 0$ had just a few closed-form solutions! Then the sum $S$ would be easy to compute. That wish is fulfilled by replacing $\tan x - x$ with a polynomial equation with simple roots. The simplest interesting polynomial is the quadratic, so experiment with a simple quadratic—for example, $x^2 - 3x + 2$.

This polynomial has two roots, $x_1 = 1$ and $x_2 = 2$; therefore $\sum x_n^{-2}$, the polynomial-root sum analog of the tangent-root sum, has two terms.

$$\sum x_n^{-2} = \frac{1}{1^2} + \frac{1}{2^2} = \frac{5}{4}. \quad (6.33)$$

This brute-force method for computing the root sum requires a solution to the quadratic equation. However, a method that can transfer to the equation $\tan x - x = 0$, which has no closed-form solution, cannot use the roots themselves. It must use only surface features of the quadratic—namely, its two coefficients $2$ and $-3$. Unfortunately, no plausible method of combining $2$ and $-3$ predicts that $\sum x_n^{-2} = 5/4$.

Where did the polynomial analogy go wrong?

The problem is that the quadratic $x^2 - 3x + 2$ is not sufficiently similar to $\tan x - x$. The quadratic has only positive roots; however, $\tan x - x$, an odd function, has symmetric positive and negative roots and has a root at $x = 0$. Indeed, the Taylor series for $\tan x$ is $x + x^3/3 + 2x^5/15 + \cdots$ (Problem 6.28); therefore,

$$\tan x - x = \frac{x^3}{3} + \frac{2x^5}{15} + \cdots. \quad (6.34)$$

The common factor of $x^3$ means that $\tan x - x$ has a triple root at $x = 0$. An analogous polynomial—here, one with a triple root at $x = 0$, a positive root, and a symmetric negative root—is $(x+2)x^3(x-2)$ or, after expansion, $x^5 - 4x^3$. The sum $\sum x_n^{-2}$ (using the positive root) contains only one term
and is simply $1/4$. This value could plausibly arise as the (negative) ratio of the last two coefficients of the polynomial.

To decide whether that pattern is a coincidence, try a richer polynomial: one with roots at $-2$, $-1$, $0$ (threefold), $1$, and $2$. One such polynomial is

$$(x + 2)(x + 1)x^3(x - 1)(x - 2) = x^7 - 5x^5 + 4x^3.$$  \hfill (6.35)

The polynomial-root sum uses only the two positive roots $1$ and $2$ and is $1/1 + 1/2^2$, which is $5/4$—the (negative) ratio of the last two coefficients. As a final test of this pattern, include $-3$ and $3$ among the roots. The resulting polynomial is

$$(x^7 - 5x^5 + 4x^3)(x + 3)(x - 3) = x^9 - 14x^7 + 49x^5 - 36x^3.$$  \hfill (6.36)

The polynomial-root sum uses the three positive roots $1$, $2$, and $3$ and is $1/1^2 + 1/2^2 + 1/3^2$, which is $49/36$—again the (negative) ratio of the last two coefficients in the expanded polynomial.

What is the origin of the pattern, and how can it be extended to $\tan x - x$?

To explain the pattern, tidy the polynomial as follows:

$$x^9 - 14x^7 + 49x^5 - 36x^3 = -36x^3 \left(1 - \frac{49}{36}x^2 + \frac{14}{36}x^4 - \frac{1}{36}x^6\right).$$  \hfill (6.37)

In this arrangement, the sum $49/36$ appears as the negative of the first interesting coefficient. Let’s generalize. Placing $k$ roots at $x = 0$ and single roots at $\pm x_1, \pm x_2, \ldots, \pm x_n$ gives the polynomial

$$Ax^k \left(1 - \frac{x^2}{x_1^2}\right) \left(1 - \frac{x^2}{x_2^2}\right) \left(1 - \frac{x^2}{x_3^2}\right) \cdots \left(1 - \frac{x^2}{x_n^2}\right),$$  \hfill (6.38)

where $A$ is a constant. When expanding the product of the factors in parentheses, the coefficient of the $x^2$ term in the expansion receives one contribution from each $x^2/x_k^2$ term in a factor. Thus, the expansion begins

$$Ax^k \left[1 - \left(\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \cdots + \frac{1}{x_n^2}\right)x^2 + \cdots\right].$$  \hfill (6.39)

The coefficient of $x^2$ in parentheses is $\sum x_n^{-2}$, which is the polynomial analog of the tangent-root sum.

Let’s apply this method to $\tan x - x$. Although it is not a polynomial, its Taylor series is like an infinite-degree polynomial. The Taylor series is
\[
\frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \cdots = \frac{x^3}{3} \left( 1 + \frac{2}{5}x^2 + \frac{17}{105}x^4 + \cdots \right). \tag{6.40}
\]

The negative of the \(x^2\) coefficient should be \(-\sum x_n^2\). For the tangent-sum problem, \(\sum x_n^2\) should therefore be \(-2/5\). Unfortunately, the sum of positive quantities cannot be negative!

**What went wrong with the analogy?**

One problem is that \(\tan x - x\) might have imaginary or complex roots whose squares contribute negative amounts to \(S\). Fortunately, all its roots are real (Problem 6.29). A harder-to-solve problem is that \(\tan x - x\) goes to infinity at finite values of \(x\), and does so infinitely often, whereas no polynomial does so even once.

The solution is to construct a function having no infinities but having the same roots as \(\tan x-x\). The infinities of \(\tan x - x\) occur where \(\tan x\) blows up, which is where \(\cos x = 0\). To remove the infinities without creating or destroying any roots, multiply \(\tan x - x\) by \(\cos x\). The polynomial-like function to expand is therefore \(\sin x - x \cos x\).

Its Taylor expansion is

\[
\left( x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \right) - \left( x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots \right). \tag{6.41}
\]

The difference of the two series is

\[
\sin x - x \cos x = \frac{x^3}{3} \left( 1 - \frac{1}{10} x^2 + \cdots \right). \tag{6.42}
\]

The \(x^3/3\) factor indicates the triple root at \(x = 0\). And there at last, as the negative of the \(x^2\) coefficient, sits our tangent-root sum \(S = 1/10\).

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**Problem 6.28**  
**Taylor series for the tangent**

Use the Taylor series for \(\sin x\) and \(\cos x\) to show that

\[
\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots. \tag{6.43}
\]

*Hint: Use taking out the big part.*
Problem 6.29  Only real roots

Show that all roots of \( \tan x - x \) are real.

Problem 6.30  Exact Basel sum

Use the polynomial analogy to evaluate the Basel sum

\[
\sum_{n=1}^{\infty} \frac{1}{n^2}.
\]

(6.44)

Compare your result with your solution to Problem 6.22.

Problem 6.31  Misleading alternative expansions

Squaring and taking the reciprocal of \( \tan x = x \) gives \( \cot^2 x = x^{-2} \); equivalently, \( \cot^2 x - x^{-2} = 0 \). Therefore, if \( x \) is a root of \( \tan x - x \), it is a root of \( \cot^2 x - x^{-2} \).

The Taylor expansion of \( \cot^2 x - x^{-2} \) is

\[
-\frac{2}{3} \left( 1 - \frac{1}{10} x^2 - \frac{1}{63} x^4 - \cdots \right).
\]

(6.45)

Because the coefficient of \( x^2 \) is \(-1/10\), the tangent-root sum \( S \)—for \( \cot x = x^{-2} \) and therefore \( \tan x = x \)—should be \( 1/10 \). As we found experimentally and analytically for \( \tan x = x \), the conclusion is correct. However, what is wrong with the reasoning?

Problem 6.32  Fourth powers of the reciprocals

The Taylor series for \( \sin x - x \cos x \) continues

\[
\frac{x^3}{3} \left( 1 - \frac{x^2}{10} + \frac{x^4}{280} - \cdots \right).
\]

(6.46)

Therefore find \( \sum x_n^{-4} \) for the positive roots of \( \tan x = x \). Check numerically that your result is plausible.

Problem 6.33  Other source equations for the roots

Find \( \sum x_n^{-2} \), where the \( x_n \) are the positive roots of \( \cos x \).

6.5  Bon voyage

I hope that you have enjoyed incorporating street-fighting methods into your problem-solving toolbox. May you find diverse opportunities to use dimensional analysis, easy cases, lumping, pictorial reasoning, taking out the big part, and analogy. As you apply the tools, you will sharpen them—and even build new tools.