Appendix 3: Derivation of the Self-Consistent Set of Equations for the Gyro-TWT

Published by

Temkin, Richard, et al.
Introduction to the Physics of Gyrotrons.

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Appendix 3

Derivation of the Self-Consistent Set of Equations for the Gyro-TWT

A self-consistent set of equations describing the gyro-TWT consists of the equations for electron motion in waveguide fields and the equation describing the wave excitation by an electron beam. For the case of electron motion in resonators, the equations for electron motion were derived in Appendix 1. It is shown there that, in general, equations for electron orbital momentum and phase (A1.4) can be written as

\[
\frac{dp_1}{dz} = -\frac{e}{v_x} \Re \left\{ CG_x e^{i\omega t} \right\}, \quad (A3.1)
\]

\[
p_\perp \frac{d\psi}{dz} = \frac{e}{v_x} \Re \left\{ CG_x e^{i\omega t} \right\}. \quad (A3.2)
\]

Also, the equation for the electron axial momentum can be written in a similar fashion as

\[
\frac{dp_z}{dz} = -\frac{e}{v_x} \Re \left\{ CG_z e^{i\omega t} \right\}. \quad (A3.3)
\]

Above, in Appendix 1, we did not consider this equation because in the case of operation at TE-modes near cutoff the axial momentum remains practically constant. These three equations correspond to the representation of the wave fields as

\[
\tilde{E} = \Re \left\{ C \tilde{E}_s (\bar{r}) e^{i\omega t} \right\}, \quad \tilde{H} = \Re \left\{ C \tilde{H}_s (\bar{r}) e^{i\omega t} \right\}.
\]

In the right-hand sides of (A3.1)–(A3.3) there are components of the function

\[
\tilde{G} = \tilde{E}_s + \left[ \frac{\bar{\nu}}{c} \times \tilde{H}_s \right].
\]

This function describes the spatial structure of the RF Lorentz force acting upon electrons. In waveguides, the electric and magnetic fields of the forward
wave can be given by
\[ E_s(\mathbf{r}) = \mathbf{e}_s(\mathbf{r}_t) e^{-ik_z z}, \quad H_s(\mathbf{r}) = \mathbf{h}_s(\mathbf{r}_t) e^{-ik_z z}, \]
so these equations can be rewritten as
\[
\frac{dp_\perp}{dz} = -\frac{e}{v_z} \text{Re}\left\{ CG_\theta e^{i(\omega t - k_z z)} \right\}, \\
\frac{d\psi}{dz} = \frac{e}{v_z} \text{Re}\left\{ CG_r e^{i(\omega t - k_z z)} \right\}, \\
\frac{dp_z}{dz} = -\frac{e}{v_z} \text{Re}\left\{ CG_z e^{i(\omega t - k_z z)} \right\},
\]
where
\[ \mathbf{\tilde{G}} = \mathbf{e}_s + \left[ \mathbf{\tilde{v}}_\perp \times \mathbf{h}_s \right]. \]

Below, the subscript \( s \), which designates the type of the operating wave, will be omitted. Components of these wave fields \( \mathbf{e}(\mathbf{r}_t) \) and \( \mathbf{h}(\mathbf{r}_t) \) were determined in Appendix 2 via the membrane function \( \Psi(\mathbf{r}_t) \) by (A2.19). This membrane function in the reference frame with the origin on the axis of electron gyration can be expanded in the Fourier series as
\[ \Psi = \sum_l \Psi_l e^{il\theta}. \]

Correspondingly, the field components are equal to
\[
\begin{align*}
\mathbf{e}_r &= \frac{k}{k_\perp^2 a} \sum_l (-i l) \Psi_l e^{-il\theta}, \\
\mathbf{e}_\theta &= -\frac{k}{k_\perp^2} \sum_l \frac{d\Psi_l}{dr} \bigg|_{r=a} e^{-il\theta}, \\
\mathbf{e}_z &= 0, \\
\mathbf{h}_\theta &= \frac{k_z}{k} \mathbf{e}_r = ne_r, \\
\mathbf{h}_r &= -\frac{k_z}{k} \mathbf{e}_\theta = -ne_\theta, \\
\mathbf{h}_z &= i \sum_l \Psi_l e^{-il\theta}. 
\end{align*}
\]
Eq. (A1.9), as is shown in Appendix 1, determines angular harmonics of the membrane function. Correspondingly, the components of the Lorentz force (A3.7) can be determined as
\[
\begin{align*}
G_r &= i \sum_l \left\{ \beta_\perp - \frac{\ell}{\kappa \xi} (1 - n_\beta z) \right\} J_j(\xi) L_1 e^{-il\theta}, \\
G_\theta &= -\frac{1}{\kappa} (1 - n_\beta z) \sum_l J'_j(\xi) L_1 e^{-il\theta}, \\
G_z &= -\frac{n}{\kappa} \beta_\perp \sum_l J'_j(\xi) L_1 e^{-il\theta}.
\end{align*}
\]
So, now we can come back to (A3.4)–(A3.6), substitute into these equations the expressions for the Lorentz force components just derived, and make an averaging of these equations over fast gyrations. Introducing a slowly varying
resonant harmonic of the gyrophase with respect to the phase of the traveling wave, \( \vartheta = s\theta - \omega t + k_z z \), these gyro-averaged equations can be written as

\[
\frac{d\varphi}{dz} = \frac{e}{\kappa v_z} (1 - n\beta_z) \text{Re} \left\{ C J'_s(\xi) L_s e^{-i\vartheta} \right\}. \tag{A3.8}
\]

\[
\frac{d\vartheta}{dz} + \frac{\omega - k_z v_z - s\Omega}{v_z} = s - \frac{e}{v_z p_\perp} \left[ \beta - s \frac{\kappa}{k} (1 - n\beta_z) \right] \text{Re} \left\{ i C J_s(\xi) L_s e^{-i\vartheta} \right\} \tag{A3.9}
\]

\[
\frac{dp_\perp}{dz} \begin{aligned}
&= n \frac{ev_\perp}{\kappa cv_z} \text{Re} \left\{ C J'_s(\xi) L_s e^{-i\vartheta} \right\}.
\end{aligned} \tag{A3.10}
\]

As one can easily see, the equation for electron energy (1.2) can be rewritten as

\[
\frac{d\xi}{dz} = -\frac{e}{v_z} (\ddot{u} \dot{E}) = -\frac{e}{v_z} v_\perp E_\theta = e \frac{v_\perp}{v_z \kappa} \text{Re} \left\{ C \sum_\xi J'_I(\xi) L_I e^{i(\omega t - k_z z - i\vartheta)} \right\}.
\]

After averaging over fast gyrations this yields

\[
\frac{d\xi}{dz} = \frac{e v_\perp}{\kappa v_z} \text{Re} \left\{ C J'_s(\xi) L_s e^{-i\vartheta} \right\}.
\]

So, again, both nonaveraged and gyroaveraged equations for the electron energy and axial momentum yield the same autoresonance integral (1.17). Correspondingly, with the use of the general relation between electron energy and total momentum, we can express the axial and orbital components of the momentum via normalized electron energy as

\[
\begin{align*}
p_z &= p_{z0}(1 - bw), \\
p_\perp &= p_{\perp0}(1 - w)^{1/2},
\end{align*} \tag{A3.11}
\]

where \( w = 2 \frac{1 - n\beta_w}{\beta_{z0}^w} \frac{\gamma - \gamma}{\gamma_0} \). This derivation is given in Sec. 7.1. As a result, below we shall consider only equations for electron energy \( w \) and slowly variable gyrophase \( \vartheta \) in which orbital and axial components of the momentum should be expressed in terms of \( w \).

Let us start from a more complicated equation for the gyrophase (A3.9). In the LHS of this equation there is a variable detuning of the cyclotron resonance. In accordance with Sec. 1.3, this detuning can be represented as a sum of two terms, the first of which determines the initial cyclotron resonance detuning at the entrance to the interaction space, while the second one describes the effect of the changes in electron energy on the cyclotron resonance condition. So,

\[
\frac{\omega - k_z v_z - s\Omega}{v_z} = \frac{\gamma_0}{v_z \gamma} \left( \omega \frac{\gamma}{\gamma_0} - k_z \frac{p_z}{m\gamma_0} - s\Omega_0 \right)
= \frac{\gamma_0}{v_z \gamma} \left[ \omega - k_z v_{z0} - s\Omega_0 - \omega (1 - n^2) \frac{\gamma_0 - \gamma}{\gamma_0} \right]
\]
(In transforming this detuning we used (A3.11).) Here we have in the RHS the ratio \( \gamma_0/v_\perp \gamma \), which is equal to \( 1/\beta_{z0}(1 - bw) \). So, finally this detuning can be represented as

\[
\frac{\omega - k_z v_z - s\Omega}{v_z} = \frac{\omega}{c\beta_{z0}(1 - bw)} \left( \frac{\omega - k_z v_z - s\Omega}{\omega} - \mu w \right), \tag{A3.12}
\]

where parameter \( \mu = \beta_{z0}^2(1 - n^2)/2(1 - n\beta_{z0}) \) characterizes the effect of the changes in electron energy on the cyclotron resonance conditions. In the RHS of (A3.9) there is an expression in square brackets, which can be rewritten as

\[
\beta_\perp - \frac{s}{k_\perp} \left( 1 - n\beta_z \right) = \frac{1}{\kappa} \left[ \kappa \beta_\perp - \frac{s(1 - n\beta_z)}{\xi} \right] = \frac{1 - n\beta_z}{s\kappa} \left( \xi - \frac{s^2}{\xi} \right). \tag{A3.13}
\]

Here we expressed the normalized orbital velocity \( \beta_\perp \) via the normalized gyroradius \( \xi \) and used the cyclotron resonance condition \( \Omega/\omega \approx (1 - n\beta_z)/s \). Now, as in Appendix I, we can use the Bessel equation, which in (A3.9) yields

\[
\left( \xi - \frac{s^2}{\xi} \right) J_s(\xi) = -\frac{d}{d\xi} \left[ \frac{dJ_s(\xi)}{d\xi} \right]. \tag{A3.14}
\]

The ratio \( (1 - n\beta_z)/\beta_z \), which appears in the RHS of (A3.9) after using (A3.13), can be rewritten as

\[
\frac{1 - n\beta_z}{\beta_z} = \frac{\gamma - np_x'}{p_x'} = \frac{1}{p_x'} \{ \gamma - \gamma_0 + \gamma_0 - n(\gamma_0 - \gamma) \}.
\]

Here the last term in square brackets, in accordance with (1.25), can be neglected. The primed axial momentum is here normalized to \( mc \). All these steps allow us to rewrite (A3.9) as

\[
\frac{d\varphi}{dz'} = \frac{1}{\beta_{z0}(1 - bw)} \left\{ \mu w - \Delta + \frac{1 - n\beta_{z0}}{\gamma_0\beta_{z0}(1 - w)^{1/2}} \text{Im} \left[ \frac{eC}{mcwK} L_s e^{-i\varphi} \right] \frac{d}{d\xi} (\xi J_s) \right\}. \tag{A3.15}
\]

Here we denoted the initial cyclotron resonance detuning \( (\omega - k_z v_{z0} - s\Omega_0)/\omega \) by \( \Delta \) and introduced the normalized axial coordinate \( z' = wz/c \).

Correspondingly, the equation for the normalized electron energy can be rewritten as

\[
\frac{dw}{dz'} = -\frac{2(1 - n\beta_{z0})}{\gamma_0\beta_{z0}\beta_{z0}} \frac{(1 - w)^{1/2}}{1 - bw} \left. \frac{J'_s(\xi)}{\xi J_s(\xi)} \text{Re} \left[ \frac{eC}{mcwK} L_s e^{-i\varphi} \right] \right\}. \tag{A3.16}
\]

Introducing a set of normalized parameters \( \mu' = \mu/\beta_{z0}, \Delta' = \Delta/\beta_{z0} \) and \( C' = eC(1 - n\beta_{z0})/mcwK\gamma_0\beta_{z0}\beta_{z0} \) allows one to reduce these equations to the
following form:
\[
\frac{d\vartheta}{dz} = \frac{1}{1 - bw} \left\{ \mu w - \Delta + \frac{1}{(1 - w)^{1/2}} \Im \left[ C L_s e^{-i\vartheta} \frac{d}{d\xi} (\xi J'_s) \right] \right\}, \tag{A3.17}
\]
\[
\frac{dw}{dz} = -2 \frac{(1-w)^{1/2}}{1-bw} J'_s(\xi) \Re \left\{ C L_s e^{-i\vartheta} \right\}. \tag{A3.18}
\]
Here primes are omitted (except for the derivative of the Bessel function).

Let us note that in the RHS of (A3.17) and (A3.18) we have the coupling coefficient \(L_s(X, Y)\), which can be different for electrons with different guiding centers. So, we can integrate these equations for a beam with different operators \(L_s\) for different beamlets and then, in the equation for wave excitation (A2.23), average the source term over the beam distribution in guiding centers. This corresponds to the integration over the cross-section area of the interaction region in the RHS of (A2.23).

Let us now consider a simple case of a cylindrical waveguide with a thin annular electron beam, in which all beamlets, in accordance with (A1.16), have the same absolute value of \(L_s\). In such a case we can introduce a new normalized amplitude of the wave,
\[
F = CJ_m(\mp k_\perp R)e^{i\Delta z}. \tag{A3.19}
\]
This reduces (A3.17) and (A3.18) to, respectively,
\[
\frac{d\vartheta}{dz} = \frac{1}{1 - bw} \left\{ \mu w + \frac{1}{(1 - w)^{1/2}} \frac{d}{d\xi} (\xi J'_s) \Im [Fe^{-i\vartheta}] \right\}, \tag{A3.20}
\]
\[
\frac{dw}{dz} = -2 \frac{(1-w)^{1/2}}{1-bw} J'_s(\xi) \Re [Fe^{-i\vartheta}]. \tag{A3.21}
\]
Note that, in accordance with the definition of the operator \(L_s\) given by (A1.16) and the amplitude \(F\) given by (A3.19), we use in (A3.20), (A3.21), and below a new slowly variable phase,
\[
\vartheta' = \vartheta + \Delta z - (s \mp m)\psi \tag{A3.22}
\]
\[
= s(\vartheta - \Omega_0 \tau) - \omega t_0 + k_z \int_0^\tau (v_z - v_{20}) d\tau' - (s \mp m)\psi.
\]
So, at the entrance, this phase has an initial value \(\vartheta'(0) = s\theta_0 - \omega t_0 - (s \mp m)\psi\), which for an unmodulated electron beam is uniformly distributed in all beamlets from 0 to \(2\pi\).

Now we can rewrite the equation for wave excitation (A2.23) in these new variables. To do this, let us use the representation of the beam current density by (A2.5) and representation of the function describing the spatial structure
of the wave electric field given by (A2.19). Then, using in (A2.23) the charge conservation law discussed in Sec. 3.3, making the Fourier transform of the membrane function (see definition of $e_0$ after (A3.7) above), and averaging this equation over fast gyration result in the following equation:

$$\frac{dC}{dz} = \frac{1}{N\kappa} \int_{S_1} j_0 \left\{ \frac{1}{\pi} \int_0^{2\pi} \frac{p_\perp}{p_z} J_s'(\xi)L_s^x e^{i\theta} d\theta \right\} ds_\perp.$$

(A3.23)

In this equation the phase $\vartheta$ is the same as that used in (A3.17)–(A3.18) and above. In the normalized variables used in (A3.17) and (A3.18) this equation can be rewritten as

$$\frac{dC}{dz} = -I_0 \int_{S_1} f(\tilde{R}_{\perp 0}) \left\{ \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - w)^{1/2}}{1 - bw} J_s'(\xi)L_s^x e^{i\theta} d\theta \right\} ds_\perp.$$

(A3.24)

Here the beam current density at the entrance was represented as $j_0 = -|I_b| f(\tilde{R}_{\perp 0})$, where the function $f(\tilde{R}_{\perp 0})$ describing the initial beam distribution over the guiding center radii is normalized to one: $\int_{S_1} f(\tilde{R}_{\perp 0}) ds_\perp = 1$ and the normalized beam current parameter $I_0$ is equal to $2(e|I_b|/mc^3)(1 - n\beta_{z0}/\gamma_0\beta_{z0}^2)^2|c^3/\omega^2 N|$, where the wave norm is given by (A2.22). So, Eqs. (A3.17), (A3.18), and (A3.24) form a self-consistent set of equations describing the gyro-TWT with an arbitrary geometry of the waveguide and the beam. Of course, when it is necessary to take into account the velocity spread in the beam, we should add in the wave excitation equation the function describing the electron velocity spread. Correspondingly, there should be an additional averaging of the source term over the velocity distribution.

In the case of a thin annular electron beam in a cylindrical waveguide, for which the equations for electron motion were reduced above to (A3.20) and (A3.21), the wave excitation equation can also be reduced further. For the normalized amplitude given by (A3.19), this equation has the following form:

$$\frac{dF}{dz} - i\Delta F = -I_0 \frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - w)^{1/2}}{1 - bw} J_s'(\xi)e^{i\theta} d\theta.$$

(A3.25)

So, the self-consistent set of equations for the axially symmetric gyro-TWT with a thin annular electron beam consists of Eqs. (A3.20), (A3.21), and (A3.25). In the last equation the normalized beam current parameter is equal to

$$I_0 = \frac{e|I_b|}{mc^3} \frac{1 - h\beta_{z0}}{\gamma_0\beta_{z0}^2} \frac{4\kappa^2}{h} \frac{J^2_{m+s}(k R_0)}{(v^2 - m^2)J^2_m(v)}.$$
In deriving this expression we used the expression for the norm of a wave in a cylindrical waveguide given by (A2.22). Note that the detuning \( \Delta \) in the LHS of (A3.25) contains the axial wavenumber, which in the presence of the wave attenuation in the waveguide walls also has the imaginary part.

For typical operating conditions we can take one more step to further simplify this set of equations. Indeed, this set of equations contains Bessel functions and their derivatives, which have the argument \( \xi = k_\perp a = \kappa(\omega/\Omega_0)\beta_{10}(1-w)^{1/2} \), which with the use of the cyclotron resonance condition can be rewritten as

\[
    \xi = \kappa \frac{s\beta_{10}}{1-n\beta_{20}} (1-w)^{1/2} = \xi_0 (1-w)^{1/2}. \tag{A3.27}
\]

This argument usually does not exceed the order of the Bessel function \( s \). Therefore, Bessel function can be represented by the polynomial \( J_s(\xi) \approx \frac{1}{s!}(\xi/2)^s \). Correspondingly,

\[
d J_s/d\xi \approx [1/(s - 1)!2^s] \xi^{s-1} \quad \text{and} \quad \frac{d}{d\xi}[\xi J'_s(\xi)] \approx [s/(s - 1)!2^s] \xi^{s-1}.
\]

So, we can now introduce the final set of normalized variables:

\[
    \varsigma = \mu z = \frac{\beta_{10}^2(1-n^2)}{2\beta_{20}(1-n\beta_{20})} \frac{\omega z}{c}, \tag{A3.28}
\]

\[
    \Delta' = \frac{\Delta}{\mu} = \frac{2(1-n\beta_{20})}{\beta_{10}^2(1-n^2)} \frac{\omega - k_\perp v_{z0} - s\Omega_0}{\omega},
\]

\[
    F' = \frac{\xi_0^{s-1}}{(s-1)!2^s} \frac{1}{\mu} F = \left( \frac{\kappa s\beta_{10}}{1-n\beta_{20}} \right)^{s-1} \frac{1}{(s-1)!2^s} \frac{2\beta_{20}(1-n\beta_{20})}{\beta_{10}^2(1-n^2)} F,
\]

\[
    l'_0 = \left[ \frac{\xi_0^{s-1}}{(s-1)!2^s} \right]^2 \frac{l_0}{\mu^2},
\]

and rewrite these equations as

\[
    \frac{d\vartheta}{d\varsigma} = \frac{1}{1-bw} \{w + s(1-w)^{(s/2)-1} \text{Im}(F e^{-i\vartheta}) \}, \tag{A3.29}
\]

\[
    \frac{dw}{d\varsigma} = -2 \frac{(1-w)^{s/2}}{1-bw} \text{Re}(F e^{-i\vartheta}), \tag{A3.30}
\]

\[
    \frac{dF}{d\varsigma} - i \Delta F = -l'_0 \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-w)^{s/2}}{1-bw} e^{i\vartheta} d\vartheta_0. \tag{A3.31}
\]

All primes are omitted here. The boundary conditions for the electron energy and phase were discussed above. The boundary condition for the forward wave amplitude at the entrance is \( F(0) = F_0 \). To establish the relation between
this amplitude and the input power, recall that in our notations the wave power in the cylindrical waveguide is equal to

\[ P = \frac{c}{8} |C|^2 \frac{1}{k^2 \kappa^4} (\nu^2 - m^2) J_m^2(\nu). \]  
(A3.32)

Here we used the field representation given after (A3.3) and the wave norm determined by (A2.22) in combination with the standard expression for the wave power in a waveguide. Now we should take into account that on the way from the amplitude \( C \) to the normalized amplitude \( F' \), which we wrote in (A3.31) without prime, we made three steps. First, after (A3.16) we introduced the normalized amplitude \( C' \) proportional to the original amplitude \( C \). Then we introduced the normalized amplitude \( F \) given in terms of \( C' \) by (A3.19), and finally we introduced the new normalized amplitude \( F' \), which was expressed in terms of \( F \) by (A3.28). So, after making all the corresponding steps, one gets the following relation:

\[ |F'| = 4 \left[ \frac{2 e^2 P G}{m^2 c^5 n} \right]^{1/2} \left[ (s\kappa)^{s-1} \frac{\beta_{10}^s}{\gamma_0 \kappa (s - 1)!} \left(1 - n\beta_{20}\right)^{3-s} \right]. \]  
(A3.33)

Here we used the coupling parameter \( G \) given by (3.59) and took into account that \( 1 - n^2 = \kappa^2 \). The ratio \( m^2 c^5 / e^2 \) in square brackets is equal to 8.687 \( \times \) 10\(^6\) kW (this is the product of \( mc^3 / e = 17.04 \) kA and \( mc^2 / e = 511 \) kV). Therefore, for the power expressed in kW, (A3.33) can be rewritten as

\[ |F'| = 1.92 \cdot 10^{-3} (PG/n)^{1/2} \left( (s\kappa)^{s-1} \frac{\beta_{10}^s}{\gamma_0 \kappa (s - 1)!} \left(1 - n\beta_{20}\right)^{3-s} \right). \]  
(A3.34)

In the case of operation at the fundamental cyclotron resonance, (A3.34) reduces to

\[ |F'| = 0.96 \cdot 10^{-3} (PG/n)^{1/2} \frac{(1 - n\beta_{20})^2}{\kappa \gamma_0 \beta_{10}^3}. \]  
(A3.35)
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