Chapter 4 Nonlinear Theory of the Gyromonotron (Single-Mode Treatment)

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CHAPTER 4

Nonlinear Theory of the Gyromonotron (Single-Mode Treatment)

4.1 Cold-Cavity Approximation

Cold-cavity approximation is the approximation we used so far. It implies that we neglect the effect of the electron beam on the axial structure of the EM field excited in an open resonator. Correspondingly, this structure is determined by the nonuniform string equation (2.13), which for axially open resonators was analyzed in detail by Vlasov et al. (1969).

The analysis of the gyrotron large-signal operation thus reduces to the integration of equations for electron motion (3.34) or (3.35) with given parameters $\Delta, \mu, \text{ and } F$ and calculation of the orbital efficiency determined by (3.36). Then, the balance equation (3.63) allows one to establish correspondence between the beam current and the EM field amplitude and, hence, translate the nonlinear dependence of the efficiency on the amplitude $F$, which describes the saturation effects, into the nonlinear dependence of the efficiency on the beam current.

Results of first calculations of the efficiency for gyrotrons with the uniform EM field ($f(\zeta) = 1$) and constant external magnetic field ($\Delta = \Delta_0$) were presented by Gaponov, Petelin, and Yulpatov (1967) for the first five cyclotron harmonics ($1 \leq s \leq 5$). It was shown that the maximum orbital efficiency slowly decreases as the harmonic number increases: this maximum efficiency is equal to 0.42, 0.29, 0.22, 0.17, and 0.14 for $s = 1, 2, 3, 4, \text{ and } 5$, respectively. These numbers were calculated for the case when all beamlets are equally coupled to the resonator EM field. Rapoport, Nemak, and Zhurakhovskiy (1967) have studied the effect of the transverse nonuniformity of this field on the efficiency at the first three harmonics. In their paper the beam interaction with the standing wave formed by a pair of mirrors was considered. (Later, this sort of gyrotron configuration was called “quasi-optical...
gyrotrons"; these devices will be discussed in Chapter 13.) It was assumed that a beam diameter is larger than a wavelength, so the transverse nonuniformity of the beam coupling to the wave is important. Rapoport, Nemak, and Zhurakhovskiy (1967) showed that this nonuniformity reduces the maximum orbital efficiency at the first three harmonics from the above-mentioned efficiencies for the case of uniform interaction 0.42, 0.29, and 0.22 to 0.31, 0.22, and 0.16, respectively.

Later, due to the reasons discussed above, attention was mostly paid to gyrotrons with the Gaussian axial distribution of the resonator field. First calculations (Moiseev, Rogacheva, and Yulpatov 1968) showed that in gyrotrons with such an axial structure of the resonator field the maximum orbital efficiency is equal to 0.72, 0.71, and 0.55 for $s = 1$, 2, and 3, respectively. Then, a more detailed parameter search was done for the first two harmonics by Nusinovich and Erm (1972) and for the first five harmonics by Danly and Temkin (1986). An example of these results is shown in Fig. 4.1. In all these studies it was assumed that the interaction starts and ends in the cross sections where the field is $e^{-3}$ times smaller than in the middle of resonator. Note also that when the interaction is restricted by cross sections where the field decays by $e$ times only, the maximum orbital efficiency is much smaller (Gryaznova, Koshevaya, and Rapoport 1969): 0.52, 0.42, and 0.38 for $s = 1$, 2, and 3, respectively. This efficiency degradation can be associated with the fact that in the latter case the electron interaction starts in a relatively strong field, while for preparation of a compact bunch it is preferable to start interaction gently, in the region of a weak field. Such a study of the effect of "tails" was done later (Gaponov et al. 1975). It was shown that the account for the cathode tail of the Gaussian distribution increases the maximum orbital efficiency up to 0.79 and 0.76 for $s = 1$ and 2, respectively, while the account for the collector tail does not play a big role.

It seems quite obvious that not only the tapering of the axial profile of the resonator field, but also the tapering of the external magnetic field can improve the interaction efficiency. First theoretical studies done by Moiseev, Rogacheva, and Yulpatov (1968) showed that for the efficiency enhancement the external magnetic field should be slightly up-tapered. Results of more detailed calculations reported by Gryaznova, Koshevaya, and Rapoport (1969) indicated that the magnetic field tapering can increase the maximum orbital efficiency at the first three cyclotron harmonics from the above-mentioned 0.52, 0.42, and 0.38 to 0.7, 0.67, and 0.53 for $s = 1$, 2 and 3, respectively. These predictions were later confirmed by theoretical studies done by other groups (Kuraev et al. 1970, Sprangle and Smith 1980, Chu, Read, and Ganguly
Fig. 4.1. Lines of equal orbital efficiencies in gyrotrons operating at the fundamental cyclotron resonance in the planes of normalized RF field amplitude, $F$, versus the normalized length, $\mu$ (a) and normalized beam current, $I_0$, versus the normalized length, $\mu$ (b). Dashed lines show optimal cyclotron resonance mismatches. Dash-dotted line $I_0 = I_{0, st}$ shows the border between the regions of soft ($I_0 > I_{0, st}$) and hard ($I_0 < I_{0, st}$) self-excitation.
1980), as well as by experiments (Glushenko, Koshevaya, and Prus 1977, Read, Chu, and Dudas 1982).

It was also shown that the tapering of the magnetic field allows one to realize the maximum efficiency in the regime of soft self-excitation (Kuraev 1979), while in the case of constant magnetic field the maximum efficiency, as is shown in Fig. 4.1, corresponds to the hard self-excitation. The difference between these two regimes is in the relationship between the beam current and the starting current. In the soft excitation regime the beam current exceeds the starting value, i.e., the self-excitation conditions are fulfilled (see Fig. 4.2a). So the oscillation amplitude can start to grow from the noise level. On the contrary, in the regime of hard excitation the starting current exceeds the beam current (see Fig. 4.2b). So the oscillations can start to grow only when the initial amplitude exceeds a certain bifurcation threshold. In practice, in order to realize the operation in the hard excitation regime, one should use some kind of start-up scenarios (Nusinovich 1974, Whaley et al. 1994). Those imply that, by varying some gyrotron parameters, we can first pass through the soft excitation region, in which the oscillations will be excited, and then reach the maximum efficiency point of destination located in the

![Graph](https://via.placeholder.com/150)

**Fig. 4.2.** Efficiency as the function of the beam current in regimes of soft (a) and hard (b) self-excitation.
Fig. 4.3. (a) Beam voltage and current rise in the case of pulsed operation; (b) efficiency as the function of the beam current for two instants of time (t₁ < t₂) in the case of proper changes in gyrotron parameters.

region of hard excitation. This statement is illustrated with Fig. 4.3. Note that, as the beam voltage increases, the relativistic cyclotron frequency decreases. Correspondingly, the cyclotron resonance mismatch ω - Ω₀, which, as we discussed in Chapter 1, should be positive for realizing the coherent cyclotron radiation, becomes larger. This makes the self-excitation harder, because only the EM field with large enough amplitude can trap electrons with substantial slippage of the gyrophase with respect to the phase of the EM field. Since there are a number of limitations on variation of gyrotron parameters (especially in pulsed operation regimes), which will be discussed later, it is obvious that it is preferable to operate in the regime of soft excitation.

4.2 Self-Consistent Approach

A self-consistent treatment of interaction between gyrating electrons and EM fields in axially open resonators implies that we include into consideration the modification of an axial structure of the field by the high-frequency component of the electron current density. This modification becomes significant when
the diffractive Q-factor is close to its minimum value, or, in other words, when the wave reflections, at least at the output, are negligibly small. (When the reflections at both ends are large enough, the axial structure is fixed, so the beam cannot modify it significantly.)

To describe the field excitation in such a case, one cannot use the field representation, which was used in Chapter 3, because in that representation the field amplitude $C_5$, determined by the beam current, was considered separately from the function $f(z)$ describing the axial structure of the field. Instead, the electric field should be represented as

$$
\tilde{E} = \text{Re}\left\{ A(z, t) \tilde{E}_s(\tilde{R}_{\perp}) e^{i\tilde{\omega}_0 t}\right\}.
$$

(4.1)

Here $\tilde{\omega}_0$ is an arbitrary chosen carrier frequency. For instance, one can choose as $\tilde{\omega}_0$ the cutoff frequency in the output cross section. Such a representation, which was, in particular, used by Ginzburg, Nusinovich, and Zavolsky (1986), allows one to describe self-consistently not only stationary but also nonstationary processes. Again, assuming that the Fresnel parameter for open resonators under consideration is large enough, one can determine the function $\tilde{E}_s(\tilde{R}_{\perp})$ by the membrane function, which obeys Helmholtz equation (AI.1), with corresponding boundary conditions. Recall (see Sec. 2.3) that in an irregular waveguide the transverse wavenumber, which is present in the Helmholtz equation, depends on $z$.

Introducing the field representation given by (4.1) into Maxwell equations and making use of the assumption about slow temporal variations in the field amplitude ($|\partial A/\partial t| << \omega |A|$) results in the parabolic equation (see, e.g., Ginzburg, Nusinovich, and Zavolsky 1986)

$$
\frac{\partial^2 A}{\partial z^2} + \frac{\omega_0^2 - \omega^2}{c^2} A - 2i \frac{\tilde{\omega}_0}{c^2} \frac{\partial A}{\partial t} = -i \frac{\tilde{\omega}_0}{cN_s} \int \tilde{S}_{\perp} \cdot \tilde{E}_s^{*} dS_{\perp}.
$$

(4.2)

Here $\omega_0 = \omega_0(z) = k_{\perp}(z)c$ is the cutoff frequency in a given cross section, i.e., the cutoff frequency of the comparison waveguide. In the right-hand side of (4.2) the norm $N_s$ is different from the norm used in Chapter 3 because we consider an irregular waveguide near cutoff. Now the norm is determined as

$$
N_s = \frac{c}{4\pi} \int_{S_{\perp}} \tilde{E}_s \cdot \tilde{E}_s^{*} dS_{\perp}.
$$

(4.3)

The integral in the right-hand side (RHS) of this equation can be reexpressed in gyrotron normalized variables, which results in the following equation

$$
\frac{\partial^2 f}{\partial \xi^2} - i \frac{\partial f}{\partial \tau} + \delta f = \frac{i_0}{2\pi} \int_{0}^{2\pi} w^{s/2} e^{i\theta} d\theta_0.
$$

(4.4)
Here we introduced the normalized time \( \tau = (\beta_{\perp 0}^4/8s^2\beta_{z 0}^2)\omega_0t \), the parameter
\( \delta = (8s^2\beta_{z 0}^2/\beta_{\perp 0}^4)[(\omega_0 - \omega_0)/\omega_0] \) characterizes the deviation of the cutoff frequency in an irregular waveguide. The variables \( w \) and \( \vartheta \) obey the same equations (3.34), in which the product \( F(\zeta) \) should be replaced by \(-2f(\zeta, \tau)\). The normalized beam current parameter \( I_0 \) in (4.4) is equal to
\[
I_0 = 64\frac{e1b^2}{mc^3} {\beta_0}^2{\beta_{\perp 0}^2} \frac{(s - 4)}{s 2} G.
\]
Eq. (4.5) is quite similar to (3.57). Here \( G \) is determined by (3.59).

Certainly, it is necessary to supplement Eq. (4.4) with the initial \( f(\tau = 0) = f_0(\xi) \) and boundary conditions. The boundary condition at the entrance, in the input cutoff cross section, can be given simply by \( f(\xi = 0) = 0 \).

At the open exit, the boundary condition in nonstationary regimes is more complicated. It can be simplified by assuming that there are no waves entering the interaction space from the output waveguide. Then, one can write for Fourier components of the field \( f_{\zeta} = \int_{0}^{\infty} f(\tau')e^{-i\Omega\tau'}d\tau' \) the condition for outgoing radiation \( \frac{d f_{\zeta}}{d\xi}|_{\zeta_{out}} = -i\xi f_{\zeta}(\zeta_{out}) \). Here \( \xi \) is the normalized axial wavenumber in the output cross section; its normalization corresponds to normalization of \( z \). Making an inverse Fourier transform of this condition results in the integro-differential equation for \( f(\zeta_{out}) \), which was derived by Ginzburg, Nusinovich, and Zavoisky (1986).

In the stationary regime, the field amplitude \( f(\zeta, \tau) \) can be represented as \( f(\zeta)e^{i\Omega\tau} \), which reduces (4.4) to the equation
\[
\frac{d^2 f}{d\xi^2} + \Omega^2 f = \frac{I_0}{2\pi} \int_{0}^{2\pi} u^{s/2}e^{i\vartheta} d\vartheta_0,
\]
with the boundary conditions \( f(0) = 0 \), \( df/d\xi|_{\zeta_{out}} = -i\sqrt{\Omega} f(\zeta_{out}) \). Here \( \Omega \)

and the normalized axial wavenumber \( \xi \) introduced above are related as \( \Omega = \xi^2 \). Certainly, the neglect of the beam effect on the field structure \( (I_0 \to 0) \) reduces (4.6) to the nonuniform string equation (2.13).

The stationary self-consistent theory developed by Bratman et al. (1973) (see also Fliflet et al. 1982) allows one to study the modification of the field structure with the variation in gyrotron parameters. It also allows one to determine optimal values of parameters for realizing the maximum efficiency. Note that the optimal parameters found in the self-consistent treatment are not very different from those determined in the cold-cavity approximation. For instance, now the optimal value of the normalized length for the operation at the fundamental cyclotron resonance is equal to 14.5 (Bratman et al. 1973), while for the fixed Gaussian profile of the field it was about 17 (see
Fig. 4.4. Nonstationary processes at positive cyclotron resonance mismatch $\Delta = 0.3$ (a), and corresponding axial structure of the module and phase of the RF field (b) and the correlation function (c).

Fig. 4.1). However, the maximum efficiency operation is realized now in the soft-excitation regime.

The nonstationary self-consistent theory (Ginzburg, Nusinovich, and Zavolsky 1986) describes the onset of oscillations, which, at relatively low currents, can be stationary oscillations with the constant amplitude. As the current increases, the device may exhibit transition from such oscillations to automodulation, and then to irregular oscillations, which can be interpreted as chaotic ones. An example of such transitions is shown in Fig. 4.4. These

Fig. 4.5. Evolution of the axial structure of the RF field with the beam current increase. Bottom figure shows corresponding dependencies of the electron orbital efficiency (solid line) and operating frequency shift (dashed line).
transitions can easily be observed in the region of backward wave operation (negative cyclotron resonance mismatch) and even in the region of small positive mismatches (gyrotron regimes with a low efficiency).

In the region of gyrotron operation with a high efficiency (large positive mismatches), the oscillations with constant amplitude, however, remain, stable even when the current significantly exceeds the optimum value. As is shown in Fig. 4.5, as the current increases, the axial structure of the EM field
experiences substantial modification, but this does not change the stability of operation. The resulting boundary, which limits the region of stationary oscillations with constant amplitude, is shown in Fig. 4.6a. Later, Airila et al. (2001) explored this parameter space in more details. The resulting map is presented in Fig. 4.6b.

Presently, such codes as MAGY (Botton et al. 1998) are available and can accurately describe self-consistent, nonstationary processes in gyrotron oscillators. One of the important results of using such codes was identification of possible parasitic excitation of various modes in the region of up-tapering in the output waveguide. In general, in a region that is not too far from the resonator exit, the cutoff frequency is still close to the operating frequency, and the magnetic field is also close enough to its resonant value. So the cyclotron resonance conditions can be fulfilled there and, hence, the interaction can continue.

4.3 Effect of Velocity Spread

As was already discussed in Chapter 2, the spread in electron axial velocities may cause the inhomogeneous Doppler broadening of the cyclotron resonance band. This, in turn, can result in the efficiency deterioration. Even in the case of operation near cutoff the difference in electron axial velocities makes a difference in electron transit times through the interaction region, which spoils the efficiency.

This effect can be illustrated with the use of Eqs. (3.24) and (3.34). As was described in Chapter 3, the equation for the slowly variable phase $\phi$ in (3.24) contains in its left-hand side (LHS) the combination $\omega/\nu_z - sh H$, which can be presented [see comments to (3.34)] as $\frac{\omega}{\nu_{z0}} \frac{\beta_z^2}{2} (\Delta + w - 1)$.

During the initial stage of electron interaction with a weak EM field near the entrance, we can assume that the energy modulation given by the last term, proportional to $w - 1$, is practically the same for all electron fractions, i.e., here $\nu_{z0} \approx \tilde{\nu}_{z0}$, where $\tilde{\nu}_{z0}$ is the mean value of the initial axial velocity. However, for the first term, which describes the electron slippage with respect to the EM field, this difference in axial velocities can be significant. So, for different fractions the difference in this term can be estimated as $\frac{\omega}{\nu_{z0}} \frac{\beta_z^2}{2} \Delta |\frac{\Delta \nu_z}{\nu_{z0}}|$. This means that the effect of energy modulation becomes visible only when this modulation $|w - 1| \sim F |\int_{\xi_{min}}^{\xi} f(\xi')d\xi'|$ is larger than $\Delta |\frac{\Delta \nu_z}{\nu_{z0}}|$. For the field with the Gaussian axial structure, $f(\xi) = \exp\left(-2\xi^2/\mu^2\right)$, this condition can be given as

$$F\mu \frac{\sqrt{\pi}}{4} (1 - \text{erf}(t)) > \Delta \left|\frac{\Delta \nu_z}{\tilde{\nu}_{z0}}\right|.$$  (4.7)
where \( \text{erf}(t) \) is the error function of the argument \( t = 2\zeta /\mu \), which is tabulated elsewhere (Abramovitz and Stegun 1964).

Using the results of efficiency calculations presented in Sec. 4.1, one can readily figure out how the spread in \( v_z \) reduces the efficiency. Recall that in that section it was mentioned that, when we take into account all the input tail of the Gaussian distribution, the maximum orbital efficiency is 0.79. Correspondingly, when we assume that the interaction starts in the cross section where the field is \( e^{-3} \) times smaller than in the maximum, the maximum orbital efficiency is 0.72, and finally, in the case when it starts at \( e^{-1} \) level, the maximum orbital efficiency is 0.52 only. Let us take for parameters \( F, \mu, \) and \( \Delta \) their optimal values: 0.14, 17, and 0.5, respectively. Then, the effect of energy modulation becomes visible only when \( 1 - \text{erf}(t) > 0.474|\Delta v_z| \). So, in the absence of the spread we can achieve the maximum orbital efficiency equal to 0.79. The axial velocity spread of about 3% makes the energy modulation visible only starting from the cross section, where the Gaussian distribution is \( e^{-3} \) times smaller than in its maximum. This reduces the maximum orbital efficiency to 0.72. Correspondingly, the effective interaction starts only in the cross section, where the field is \( e^{-1} \) times smaller than in the maximum, when the axial spread is close to 33%. Note that in typical electron beams formed by the magnetron injection guns the axial velocity spread is on the order of a few percentages. Therefore, our simple consideration can be used as an explanation to the fact that for many experiments the results of calculations done by Nusinovich and Erm (1972) and Danly and Temkin (1986) were in reasonable agreement with experimental data. (Recall that in those studies it was assumed that the interaction space is limited by the cross sections where the field is at \( e^{-3} \) level.)

Needless to say that the velocity spread also restricts the maximum orbital-to-axial velocity ratio, which can be used in gyrotrons. Since typically just the energy of orbital motion is extracted in the process of interaction in gyrotrons, it is desirable to make this ratio as large as possible. However, in the case of significant spread, electrons with initially small axial velocities will be then reflected in the magnetic mirror, formed by the increasing magnetic field. This can lead to the formation of a cloud of trapped electrons in the region between the gun and the resonator.

The effect of velocity spread on the efficiency was studied in a large number of papers. One of the first and quite general studies of this effect has been performed by Ergakov, Moiseev, and Erm (1980). It was assumed there that the axial structure of the resonator field can again be described by the Gaussian distribution and also that all electrons have the same kinetic energy, but a
certain spread in orbital-to-axial velocity ratios. The latter was described by the Gaussian distribution in orbital velocities

\[ f_e(v_{\perp}) \sim \exp \left\{ -4 \left( \frac{v_{\perp,0} - \bar{v}_{\perp,0}}{\Delta v_{\perp}} \right)^2 \right\}. \]  

(4.8)

Note that the spread $\Delta v_{\perp}$ at $e^{-1}$ level in (4.8) is $2\sqrt{2}$ times larger than the RMS value. Some special means were undertaken in order to eliminate the electrons with too large orbital velocities from consideration, since such mirrored particles cause singularities in the equations of motion written in Lagrangian coordinates.

Ergakov, Moiseev, and Erm (1980) have optimized the gyrotron parameters for maximizing the total electron efficiency

\[ \eta = \frac{1}{2 \left( 1 - \gamma_0^{-1} \right)} \int v_{z0} \beta_{10}^2 f_e(v_{\perp,0}) \eta_{\perp,0} dv_{\perp,0}. \]  

(4.9)

Before presenting their results, let us make some comments to (4.9). This equation is a generalization of (3.37) for the case of a beam with velocity spread. In such a beam, a number of electrons with orbital velocities in the interval $\Delta v_{\perp}$ is proportional to $\Delta n \sim f_e(v_{\perp,0}) \Delta v_{\perp}$. The microwave power extracted from this fraction of the beam is $\eta V_b \Delta I_b$. Here the efficiency $\eta$ is determined by (3.37) and the current is given by $\Delta I_b = v_z \Delta \rho = ev_z \Delta n$. So, this power can be determined as

\[ \Delta P = \frac{\beta_{10}^2}{2 \left( 1 - \gamma_0^{-1} \right)} \eta_{\perp,0} eV_b v_z f_e(v_{\perp,0}) \Delta v_{\perp}. \]

Correspondingly, the total power extracted from the beam is

\[ P = \frac{eV_b}{2 \left( 1 - \gamma_0^{-1} \right)} \int \beta_{10}^2 \eta_{\perp,0} v_{z0} f_e(v_{\perp}) dv_{\perp}. \]  

(4.10)

The total efficiency is the ratio of this power to the beam power. In (4.10) the distribution function is normalized as $\int_{v_{\perp}} f_{v_{\perp}} f_e(v_{\perp}) v_z dv_{\perp} = 1$. This implies $f_{v_{\perp}}(v_{\perp}) = e/I_b f_e(v_{\perp})$. Thus, using (4.10) yields (4.9).

Some results of simulations are presented in Fig. 4.7, which shows the evolution of optimal parameters with the increase in the velocity spread. The results shown look quite optimistic. They predict that even in the case of the 60% spread, which is equivalent to more than 20% RMS spread in orbital velocities, the maximum efficiency can be about 30%. The optimal length gets smaller, as the velocity spread increases. This tendency can be associated with the increasing role of slowly moving electrons, which interact with the field.
for a rather long time. Correspondingly, as the spread increases, the efficiencies achievable in the regime of soft self-excitation get closer to the maximum efficiency, as is shown by Ergakov, Moiseev, and Erm (1980). An interesting finding from their study was the conclusion that in the range of spreads $0.1 < \delta v_\perp < 0.8$ (here $\delta v_\perp = \Delta v_\perp / \bar{v}_\perp$) the optimal value of the pitch-ratio $\alpha = \bar{v}_{\perp 0} / \bar{v}_z$ approximately obeys the relation $\alpha^2 \delta v_\perp \approx 1$. This makes experimentally achievable $\alpha$'s ($\alpha \leq 2$), in the presence of the velocity spread, closer to their optimal values.

### 4.4 Space-Charge Effects

As in any microwave source driven by an electron beam, in the gyrotron one can distinguish two kinds of space charge effects, which are caused, respectively, by DC and AC space-charge fields.

The voltage depression caused by the beam DC space charge was considered in Sec. 2.2 (see also Drobot and Kim 1981). In addition to that consideration, let us note that in real devices the vacuum is typically at the level $10^{-7} - 10^{-8}$ Torr. This means that there are always some residual gases, which can be ionized by the beam impact ionization. The presence of these ions can compensate, to some extent, for the DC space charge force. Let us also note that when the beam has a finite thickness, of course, the voltage depression, which varies across the beam, causes a certain spread in electron velocities.
The role of AC space charge effects is more specific. Its nature is similar to the negative mass instability (Nielsen and Sessler 1959, Kolomenskiy and A. N. Lebedev 1959), which was briefly mentioned in Chapter 1. These effects can be present not only in the interaction regions, but also in regions free from microwaves (like drift regions of gyrokystrons). Therefore, it makes sense to explain the nature of these effects by considering the electrons in such drift tubes. Such a treatment was, first, done by V. K. Yulpatov (1970), who, however, did not publish his results. Later, Bratman (1976), Symons and Jory (1981), and Charbit, Herscovici, and Mourier (1981) considered the same problem by using slightly different approaches. Since, typically, the beam radius is much larger than the wavelength and the gyroradius of electrons, a thin annular electron beam can be replaced by a thin electron layer, which is schematically depicted in Fig. 4.8. In our explanation of the effect we will follow Bratman (1976) and Symons and Jory (1981).

A gyrating electron experiences the action of the space-charge force, which results from the contributions of all other electrons. Certainly, in an unperturbed beam, this force is zero in the midplane of a beam and has a maximum on the top of the layer and minimum in the bottom. The mean value of this force is equal to zero.

Assume now that for some reason (like a bunching caused by initial modulation of electron energies in the input resonator) there is a region of highest space-charge density, which is shadowed in Fig. 4.8. This layer of the highest density oscillates with the relativistic electron cyclotron frequency. Then, a test particle, A, shown in the figure, which is moving ahead of the bunch, will be outside this shadowed region during its journey up, and thus will be accelerated by the space charge field of the layer. On its way down, this particle will be decelerated. However, particle A will spend a smaller time on its way back because this layer was also moved up during this time interval. Therefore, there will be a net acceleration of this electron. Correspondingly, as a result, it will start to gyrate slower and its orbit increases, as is shown by...
a small arrow in Fig. 4.8. Likewise, an electron, B, gyrating behind the bunch will be after all decelerated by the bunch, and hence, will gyrate faster and execute smaller orbits, as shown in the figure. So, as a result, the beam space charge increases the density of electrons in the layer, and hence, enhances the electron bunching.

In gyroklystrons, this effect can be used for shortening the drift sections. The AC space charge fields can also be important in the regions between the electron gun and the interaction space, because they may cause some space charge instabilities. As was shown by Liu, Antonsen, and Levush (1996), the growth rate of these instabilities is proportional to \( \omega_p^2 / \Omega^2 \), where \( \omega_p \) is the beam plasma frequency. Since such instabilities, once they occur in the region of beam formation, can induce the spread in electron velocities and energies, they were actively studied for the case of electron beams propagating in adiabatically increasing external magnetic fields (see Liu, Antonsen, and Levush 1996 and references therein).

As we just noted, the role of space-charge effects in gyrotrons is characterized by the ratio \( \omega_p^2 / \Omega^2 \). For practical purposes it makes sense to show how this ratio can be expressed in terms of the beam current and beam geometry. Let us start from the standard definition of the electron density,

\[
\nu = \frac{e l_b}{m c^3} \frac{1}{\beta_c r_c S_b},
\]

where \( r_c = e^2 / mc^2 \approx 2.82 \cdot 10^{-13} \text{cm} \) is the classical electron radius. For the beam current expressed in Amperes, (4.11) can be rewritten as

\[
\nu (\text{cm}^{-3}) = 0.21 \cdot 10^9 \frac{l_b(A)}{\beta_c S_b(\text{cm}^2)}.
\]

Correspondingly, the ratio of squared frequencies can be determined as

\[
\frac{\omega_p^2}{\omega^2} = \frac{4\pi e^2 n}{m \gamma_0 \omega^2} = \frac{e l_b}{m c^3} \frac{1}{\beta_c \gamma_0} \frac{\lambda^2}{\pi S_b}.
\]

When an electron beam can be treated as a thin annular electron beam with the guiding center radius much larger than the Larmor radius, \( S_b \approx 4\pi R_b \nu_\perp / \Omega_0 \). Then, taking into account the cyclotron resonance condition, (4.13) can be rewritten as

\[
\frac{\omega_p^2}{\omega^2} = \frac{e l_b}{m c^3} \frac{1}{s \beta_0 \beta_\perp \gamma_0} \frac{\lambda}{2\pi R_b}.
\]
The ratio $\omega_p^2/\Omega^2$ has a quite similar form: the only difference is in the place of the cyclotron harmonic number $s$: instead of the denominator in (4.14) it should be placed in the numerator.

The AC space charge effects, as was shown by Bratman and Petelin (1975), slightly decrease the interaction efficiency in gyromonotrons. In their paper it was shown that the dependence of the efficiency on the beam space charge can be approximated by the formula

$$\eta = \eta(S = 0) + S \frac{\partial \eta}{\partial S} \bigg|_{S=0}. \quad (4.15)$$

Here the space charge parameter $S$ is equal to $(4/\pi B_\perp)\omega_p^2/\Omega^2$ and the absolute value of the negative derivative $\partial \eta/\partial S$ is on the order of one. A typical example was also studied, for which it was shown that the space-charge effects reduce the efficiency from 42% to 38%.

### 4.5 Trade-Offs in the Gyrotron Design

Above, we described some methods of optimizing the gyrotron parameters for maximizing the interaction efficiency. However, in the design of gyrotrons not only interaction efficiency but also other operating parameters, which characterize the device performance, are of great interest. Therefore, typically, researchers and designers deal with some trade-offs, part of which will be discussed below. Here, we will discuss the output efficiency and the output power. The power limits in the peak pulse and CW or repetitive-pulse operations will be considered separately.

Output efficiency is the ratio of the power radiated from the resonator to the beam power. Recall that there are two sorts of losses: diffractive losses, which cause the outgoing radiation, and Ohmic losses, which cause the power dissipation in the resonator walls. Correspondingly, the output efficiency, $\eta_{out}$, relates to the interaction or electronic efficiency, $\eta_{el}$, which we analyzed so far, as

$$\eta_{out} = \left(1 - \frac{Q}{Q_{ohm}}\right)\eta_{el} = \frac{Q_{ohm}}{Q_{ohm} + Q_D} \eta_{el}, \quad (4.16)$$

where the total $Q$ is determined by (3.41). Clearly, when $Q_{ohm} >> Q_D$, the Ohmic losses are negligibly small in comparison with diffractive losses, and $\eta_{out} \approx \eta_{el}$.

As one can easily estimate by using Eqs. (3.42) and (3.44), at wavelengths in the range of centimeters and long millimeters $Q_{ohm} >> Q_D$, so there is
no need to distinguish these two efficiencies. However, at short millimeters and especially at submillimeters, the ohmic $Q$ decreases rapidly. Also, in some cases, as the wavelength shortens, the interaction cross section and the beam current get smaller, which makes it necessary to increase the diffractive $Q$ in order to be able to excite oscillations with a reasonable efficiency. In such cases one should optimize the output efficiency instead of the electronic efficiency, or, in other words, to find a trade-off between Ohmic losses and interaction efficiency. This adds one more parameter, the ratio of diffractive to ohmic $Q$-factors, to the set of normalized parameters that characterize the gyrotron efficiency. V. S. Ergakov, M. A. Moiseev, and V. I. Khizhnyak carried out detailed optimization of gyrotrons operating at the first two cyclotron harmonics with the account for ohmic losses. Their unpublished results were presented later in the review paper (Nusinovich and Pankratova 1981). These results demonstrated gradual reduction in the output efficiency with the increase in ohmic losses. Temkin et al. (1979) treated this problem in a similar way.

In the limit of small currents, it makes sense to increase the interaction length in order to get a substantial bunching in a weak EM field. In such a case the optimal value of the ratio $Q_D/Q_{ohm}$, as is shown by Nusinovich and Pankratova (1981), is $\frac{1}{2}$. A much more favorable scaling for the efficiency occurs in the case of corresponding lengthening of the interaction space, which was discussed in Sec. 3.2. (In more detail, this scaling is discussed by Nusinovich 1984.) Recall that the electron motion and the efficiency in such a case can be described by Eqs. (3.38) and (3.39), respectively.

**Peak output power**, as follows from the balance equation (1.30), is equal to

$$P_{out} = \eta_{out} V_b I_b = \frac{\omega}{Q_D} W.$$  \hspace{1cm} (4.17)

So, to increase this power, one should increase the beam power and try to keep the efficiency unchanged. The latter is not so easy because the orbital efficiency depends on the normalized interaction length and amplitude of the EM field. This length, in turn, determines the diffractive $Q$, while the amplitude determines the microwave energy stored in the resonator [see RHS of Eq. (4.17)]. Clearly, as the power increases, the field amplitude increases as well. As follows from Fig. 4.1a, electron overbunching in EM fields that are too strong can be to some extent avoided, when the field amplitude increase is accompanied with the shortening of the interaction length. These changes in length will also decrease the diffractive $Q$. Consistent changes in the field amplitude and interaction length allow designers to combine a substantial power enhancement with a slight efficiency degradation.
Average power is limited by microwave ohmic losses in a cavity. Mean value of the density of these losses can be estimated as

$$\bar{\rho}_{\text{ohm}} = \frac{Q_D}{Q_{\text{ohm}}} \frac{P_{\text{out}}}{S}. \quad (4.18)$$

where $S = 2\pi RL$ is the wall area in an open resonator. Taking into account definitions of $Q_D$ and $Q_{\text{ohm}}$ given above, one can readily show (see, e.g., Flyagin and Nusinovich 1988) that

$$\bar{\rho}_{\text{ohm}} \propto \frac{P_{\text{out}}}{\lambda^2} \frac{L \delta_{sk}}{R^2 (1 - \frac{m^2}{\nu^2})}. \quad (4.19)$$

Since this density of ohmic losses is limited by cooling capabilities and typically does not exceed $\bar{\rho}_{\text{ohm}}^{\text{max}} = 1 - 2kW/cm^2$, (4.19) sets the limit for the maximum output power in such regimes. Let us point out that this limit can play an important role even when $Q_{\text{ohm}} >> Q_D$. With the account for the dependence $\delta_{sk} \propto \sqrt{\lambda}$ given by Eq. (3.43), Eq. (4.19) shows that when resonator dimensions scale linearly with the wavelength ($L/\lambda$, $R/\lambda = \text{const}$), which implies operation at a given mode in a resonator with a given diffractive $Q$, the maximum average power scales as $P_{\text{out}}^{\text{max}} \propto \lambda^{5/2}$. This is a standard scaling law for various microwave tubes. Alternatively, Eq. (4.19) shows how to expand the cross section of the interaction region in order to keep the output power constant. It follows from (4.19) that to realize this, when $L/\lambda = \text{const}$, the resonator radius should scale as $R \propto \lambda^{-1/4}$. Certainly, this means the shift of operation into the region of high-order modes with a very dense spectrum, where simultaneous interaction with several modes becomes possible.

### 4.6 Problems and Solutions

**Problems**

1. By using assumptions made in Sec. 4.2, derive from the wave equation the parabolic equation (4.2).
2. Determine the mean orbital-to-axial velocity ratio, $\alpha = \bar{v}_\perp / \bar{v}_z$, at which 1% of electrons in a beam with the Gaussian velocity distribution function given by (4.8), is reflected (mirrored) in the magnetic bottle. Assume the orbital velocity spread to be equal to 10%.
3. Determine the residual gas density at the pressure $10^{-7}$ Torr and compare it with the electron density in a $40\,\text{A}$, $80\,\text{kV}$ beam with the orbital-to-axial velocity ratio 1.5. Assume that the mean value of the beam radius is 1 cm and the spread in guiding center radii can be neglected. Also assume
that the external magnetic field value corresponds to the operation at the 100 GHz frequency at the fundamental cyclotron resonance.

4. Evaluate the space-charge parameter $S$ determined in Sec. 4.4 for the case considered in Problem 3 neglecting the effect of residual gas.

5. Assuming the maximum density of ohmic losses in a copper cavity $p_{\text{ohm}}^{\text{max}} = 1\text{ kW/cm}^2$ and $L/\lambda = 7$, estimate the maximum CW output power of a gyrotron with the minimum $Q_{\text{diff}}$ cavity operating in the TE$_{0,3}$-mode at 100 GHz.

**Solutions**

1. Let us start from the wave equation written as

$$\Delta \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = -\frac{4\pi}{c^2} \frac{\partial \vec{j}}{\partial t}.$$ 

Taking into account that we consider the fields excited near cutoff frequencies, let us represent the electric field and the electron current density, respectively, as

$$\vec{E} = \text{Re}\left\{ A(z, t) \vec{E}_s(r_\perp)e^{i\omega_0 t}\right\}, \quad \vec{j} = \text{Re}\left\{ j_\omega e^{i\omega_0 t}\right\}.$$ 

Here the field amplitude is a slowly variable function of time ($|\partial A/\partial t| \ll \omega_0 |A|$), $\omega_0$ is an arbitrarily chosen carrier frequency, which can be the cutoff frequency in a certain cross section, and the function describing the transverse distribution of the electric field obeys the known equation $\Delta_\perp \vec{E}_s + k_\perp^2 \vec{E}_s = 0$. In this equation, $k_\perp(z) = \omega_0(z)/c$ and $\omega_0(z)$ is the cutoff frequency in a given cross section, i.e., the cutoff frequency of the comparison waveguide. The assumption about slow variation of the field amplitude allows us to ignore the second time derivative in the wave equation, i.e., to write $\frac{\partial^2 \vec{E}}{\partial t^2} \simeq \text{Re}\left\{ \vec{E}_s(r_\perp)(-\omega_0^2 A + 2i\omega_0 \frac{\partial A}{\partial t})e^{i\omega_0 t}\right\}$. Also, in the right-hand side of this equation, we can represent the time derivative as $\frac{\partial \vec{j}}{\partial t} = \text{Re}\{i\omega_0 j_\omega e^{i\omega_0 t}\}$.

Substituting these representations into the wave equation, multiplying both sides of this equation by $\vec{E}_s^\star$, and integrating over the cross-section area results in the parabolic equation (4.2). Note that, in general, the electric field can be represented as a superposition of eigenmodes; however, when these modes are orthogonal, the result is the same, because integration over the cross section leaves only one mode of choice from the superposition.

2. As follows from the definition of the velocity distribution function given by (4.8), there will be 1% of reflected particles when the error function
erf\(t_0\) is equal to 0.99, and hence, the argument \(t_0 = 2(v_0 - \bar{v}_{\perp 0})/\Delta v_{\perp 0}\) is equal to 0.01. Here \(v_0\) is the total velocity of all electrons. Thus, when the orbital velocity spread is equal to 10%, this value of the argument \(t_0\) corresponds to the ratio of the total electron velocity to the mean value of the orbital velocity equal to 1.091. Using known expressions for the total and orbital velocities [cf. (1.19) and (1.21)], one can readily find that this ratio corresponds to the mean orbital-to-axial velocity ratio close to 2.29.

3. Using the known relation between the pressure and density, one can easily find that the residual gas pressure \(10^{-7}\) Torr corresponds to the density of the residual gas close to \(3.55 \times 10^9\) cm\(^{-3}\). Then, to determine the beam density, one should use (4.11). The Larmor radius of electrons for given beam parameters is equal to 0.02 cm, hence the beam area is close to 0.25 cm\(^2\). Substituting this and other given parameters into (4.11) yields for the beam density the value close to \(1.2 \times 10^{11}\) cm\(^{-3}\). So, the beam density is almost two orders of magnitude higher than the density of a residual gas and, therefore, one should expect a certain voltage depression caused by the beam space charge during the initial stage of a long pulse. (The duration of this stage depends on the ionization cross sections of the gases present as well as on other parameters determining the ionization time.)

4. Using the well-known relation between the electron density and the electron plasma frequency and assuming that the cyclotron frequency is close to the operating frequency (100 GHz), one can easily find that the squared ratio of these frequencies is approximately equal to \(10^{-3}\). Hence, the space charge parameter \(S\) is close to \(7 \times 10^{-2}\).

5. The maximum power can be found from (4.18). As follows from (3.43) and (3.44), the Ohmic Q-factor for the case of an ideal copper is close to \(2.3 \times 10^4\). (Here we estimated the cavity wall radius assuming that the operating frequency is close to the cutoff frequency, and therefore \(R_w \approx 0.486\) cm.) Then, the minimum diffractive Q-factor, as follows from (2.11), is equal to 616. Hence, the maximum CW power is equal to 237 kW. Note that in the reality, first, the skin-depth is larger than its theoretical value, as we discussed in Chapter 3, and second, the diffractive Q-factor is larger than its minimum value, as shown in Fig. 2.7. Therefore, in real devices, the maximum CW power, which can be achieved in the case of operation at the TE\(_{0,3}\)-mode at 100 GHz, is lower than this number.