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THE MATHEMATICAL LEGACY OF RICHARD TIMONEY

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Introduction

In this article we survey and comment on the mathematical publications of Richard Martin Timoney. It is, of course, impossible to summarise in one article the full extent of forty productive years of mathematical research and, at the same time, include all the necessary background information. This is particularly true of Richard, who employed ideas and techniques from many different areas in his research. Indeed, his thesis involved one-variable complex analysis, several complex variables, harmonic functions, complex differential geometry, set theory, topology, geometry of Banach spaces, tensor products, Lie algebras and Lie groups. This suited Richard. In one of his early publications he says: *One of the reasons that Bloch functions are interesting is that they can be characterised in many different ways and, so, they arise in many different contexts* (see Theorem 3 below). As he moved into a new area, Richard brought with him and made effective use of ideas and techniques from his previous areas. Indeed, one of the most effective tools he devised in studying elementary operators was the *tracial geometric mean* (see Theorem 24 below), and when he discovered later that it was already in use in quantum information theory under the name *fidelity* he wrote (see [32]): *One might hope that the different perspectives in operator theory and quantum information theory would lead to new results in one area inspired by the other*. His own research is a tribute to this philosophy and was the reason that the conference in Richard's honour in May 2019 was called **A**(nalysis), **G**(eometry), **A**(lgebra).

Richard wrote concisely and with clarity, and sought to make his mathematics accessible, but he was also aware that his readers might not have his extensive mathematical culture. In one of his papers he writes: *The notation $H_z(u, \bar{v})$ is not a standard one. It is used here so as to avoid giving the impression that the reader needs familiarity with differential geometry*. We have been selective in our

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presentation and generally opted for results that could be presented without extensive preliminary results, or whose background is generally accepted as being well known. In our attempt to make this article self-contained we have, on occasion, for instance in defining the spectral radius, taken the shortest rather than the standard approach. Richard was always scrupulous in crediting results to their authors. It proved impossible to do likewise in this survey, and we refer the reader to Richard's papers for the appropriate citations.

We have divided Richard's publications into five groups: Bloch functions, infinite-dimensional holomorphy, elementary operators, operator theory and miscellaneous papers; his publications have been divided into groups corresponding to these divisions. A bibliography of other articles mentioned is also included. We label results from Richard's papers as theorems, while other results are presented as propositions. Richard wrote three survey articles on his own research [11, 15, 39], and the articles he wrote to popularise mathematics are listed in the obituary [D8] that appeared in the Bulletin of the Irish Mathematical Society.

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Notation

All linear spaces are over the complex numbers $\mathbb{C}$. Let $\Delta := \{z \in \mathbb{C} : |z| < 1\}$ denote the unit disc in the complex plane with boundary the unit circle $\mathbb{T}$. For each positive integer m , let $\mathbb{C}^m$ denote an m -dimensional Hilbert space with unit ball B_m . If X and Y are locally convex spaces, let $\mathcal{L}(X, Y)$ denote the set of all continuous linear mappings from X to Y . We let $X' := \mathcal{L}(X, \mathbb{C})$ when X is a Banach space, write A^* in place of $\mathcal{L}(A, \mathbb{C})$ when A is a Banach algebra, and let $\mathcal{B}(X) := \mathcal{L}(X, X)$. If $(X, \|\cdot\|)$ is a Banach space, $x \in X$ and $r > 0$, we let $B_r(x) := \{y \in X : \|y - x\| < r\}$ and let $B_X = B_1(0)$.

An operator $T \in \mathcal{B}(X)$ is *Hermitian* if

$$\|e^{itT}(x)\| = \left\| \sum_{n=0}^{\infty} \frac{(itT)^n(x)}{n!} \right\| = \|x\|$$

for all $x \in X$ and all $t \in \mathbb{R}$, or, equivalently, if T has *real numerical range*, that is, if

$$V(T) := \{\varphi(Tx) : (x, \varphi) \in X \times X', \|\varphi\| = \|x\| = \varphi(x) = 1\} \subset \mathbb{R}$$

(see [B7, p. 84]). A Banach algebra A with involution $*$ such that $\|xx^*\| = \|x\|^2$ for all $x \in A$ is a $\mathcal{C}^*$ -algebra. An element x in a $\mathcal{C}^*$ -algebra A is *self-adjoint*, that is $x = x^*$, if and only if it is Hermitian, and $x \in A$ is *positive* if and only if $x = yy^*$ for some $y \in A$. If A is a Banach algebra, then $x \in A$ is positive if $V(T) \subset \mathbb{R}^+ = [0, \infty)$. An operator T between Banach algebras is positive if it maps positive elements to positive elements. If $x \in A$ the spectral radius of x , $r_\sigma(x)$, equals $\lim_{n \rightarrow \infty} \|x^n\|^{1/n}$.

The space $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, plays a number of different important roles in Richard's research. If $\mathcal{H}$ is n -dimensional, then $\mathcal{B}(\mathcal{H})$ can be identified with $M_n(\mathbb{C})$, the space of $n \times n$ complex matrices. Composition of operators defines multiplication, and the adjoint T^* defines an involution where T^* is the unique operator satisfying $\langle Tx, y \rangle = \langle x, T^*y \rangle$. We let $\xi \in \mathcal{H} \longrightarrow \bar{\xi} \in \mathcal{H}$ denote a conjugation. A transposition $x \longrightarrow x^t$ is defined on $\mathcal{B}(\mathcal{H})$ by letting $x^t(\xi) = \overline{x^*(\bar{\xi})}$. The space $\mathcal{B}(\mathcal{H})$ is a Cartan factor, a C^* -algebra, a *von Neumann algebra* (that is a C^* -algebra isometrically isomorphic to a dual Banach space), a JB^* -triple, an operator space and more. Moreover, every C^* -algebra can be realised as a $*$ -closed Banach subalgebra of $\mathcal{B}(\mathcal{H})$ and a JB^* -triple with no exceptional part, that is, a JC^* -algebra can be realised as a Banach subspace of $\mathcal{B}(\mathcal{H})$ closed with respect to the triple product $\{a, b, c\} \longrightarrow (ab^*c + cb^*a)/2$. A *representation* of a C^* -algebra A is a pair $(\mathcal{H}, \pi)$ where $\mathcal{H}$ is a Hilbert space and π is a $*$ -homomorphism from A into $\mathcal{B}(\mathcal{H})$.

A sequence of finite-dimensional subspaces $(X_n)_{n=1}^\infty$ of X is a *finite-dimensional decomposition* if for each $x \in X$ there is a unique sequence $(x_n)_{n=1}^\infty, x_n \in X_n$ for all n , such that $x = \sum_{n=1}^\infty x_n =: \lim_{m \rightarrow \infty} \sum_{n=1}^m x_n$. If each X_n is one-dimensional and e_n spans X_n , then $x_n = \alpha_n e_n$ where $\alpha_n \in \mathbb{C}$ and $(e_n)_{n=1}^\infty$ is a basis (or Schauder basis) for X . If $(X, \|\cdot\|)$ is a Banach space with finite-dimensional decomposition $(X_n)_{n=1}^\infty$, and for $x_n \in X_n$

$$\left\| \sum_{n=1}^\infty x_n \right\| = \sup \left\{ \left\| \sum_{n=1}^\infty \lambda_n x_n \right\| : |\lambda_n| \leq 1, n = 1, 2, \dots \right\},$$

then the decomposition is *1-unconditional*.

If $\mathcal{D}$ is a domain (a connected open set) in X , then $f : \mathcal{D} \longrightarrow Y$ is *holomorphic* if f is continuous and for each $a \in \mathcal{D}$, $x \in X$ and $\varphi \in Y'$ the mapping

$$\lambda \longrightarrow \varphi \circ f(a + \lambda x)$$

is holomorphic as a function of one complex variable in a neighbourhood of the origin (see [D7, chapter 3] and [M4] for a number of equivalent definitions). Let $\mathcal{H}(\mathcal{D}, Y)$ denote the vector space of holomorphic mappings from $\mathcal{D}$ to Y , endowed, unless otherwise stated, with the compact open topology τ_0 . We write $\mathcal{H}(\mathcal{D})$ in place of $\mathcal{H}(\mathcal{D}, \mathbb{C})$ and let $\mathcal{H}^\infty(\mathcal{D}) := \{f \in \mathcal{H}(\mathcal{D}) : f(\mathcal{D}) \text{ is bounded in } \mathbb{C}\}$. A bijective mapping $\phi \in \mathcal{H}(\mathcal{D}_1, \mathcal{D}_2)$ is *biholomorphic* if $\phi^{-1} \in \mathcal{H}(\mathcal{D}_2, \mathcal{D}_1)$. Bijective holomorphic mappings are biholomorphic when the domain and range lie in finite-dimensional spaces, but it is an open problem if this result extends to infinite-dimensional Banach spaces. When $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}$, the group of biholomorphic mappings from $\mathcal{D}$ to itself is denoted by $\text{Aut}(\mathcal{D})$.

A domain $\mathcal{D}$ is *homogeneous* if for any pair $x, y \in \mathcal{D}$ there exists $\phi \in \text{Aut}(\mathcal{D})$ such that $\phi(x) = y$. A domain $\mathcal{D}$ is *symmetric* if for all $x \in \mathcal{D}$ there exists $\varphi_x \in \text{Aut}(\mathcal{D})$ such that $\varphi_x^2(z) = z$ for all $z \in \mathcal{D}$, and x is the unique fixed point of φ_x (we call φ_x a symmetry about x). A symmetric (respectively homogeneous) domain that is holomorphically equivalent to a bounded domain is called a *bounded symmetric*

domain (respectively a *bounded homogeneous domain*). Every bounded symmetric domain is homogeneous, and a homogeneous domain that admits a symmetry at one point is symmetric. In particular, the symmetry $\{z \rightarrow -z\}$ shows that the open unit ball of a Banach space is symmetric if and only if it is homogeneous. The collection of bounded homogeneous domains is much larger than the collection of bounded symmetric domains. For example, in every finite-dimensional space there are only a finite number of non-holomorphically equivalent bounded symmetric domains, but, apart from a few spaces of small dimension, there are in every finite-dimensional space an uncountable number of non-equivalent bounded homogeneous domains. The theory of bounded symmetric domains is also very well understood; that of bounded homogeneous domains not so well.

The irreducible finite-dimensional bounded symmetric domains have been classified and, apart from two exceptional domains in dimension 16 and 27, are bi-holomorphically equivalent to the unit balls of the Cartan factors.¹ The *Cartan factors* are (i) the $n \times m$ matrices, (ii) the $n \times n$ symmetric matrices, (iii) the $n \times n$ skew symmetric matrices and (iv) the finite-dimensional spin factors. Moreover, any finite-dimensional bounded symmetric domain is biholomorphically equivalent to a finite product of irreducible bounded symmetric domains, unique up to permutation of factors (Cartan's Theorem). This allows, in the absence of a general proof, a case-by-case examination of the irreducible domains, and then hopefully a method of combining these proofs to obtain a general proof. Moreover, the infinite analogues are also available (see [H5]) and, in particular, if H_1 and $\mathcal{H}_2$ are Hilbert spaces, then the open unit ball of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ is a bounded symmetric domain.

Bounded symmetric domains have been characterised as the open unit balls of a collection of Banach spaces, the JB^* -triples.

Definition 1. A JB^* -triple is a pair $(X, \{\cdot, \cdot, \cdot\})$ where X is a Banach space and the triple product $\{\cdot, \cdot, \cdot\} : X^3 \rightarrow X$, $(x, y, z) \rightarrow \{x, y, z\}$ is linear in x , symmetric in x and z , anti-linear in y , and satisfies the following axioms:

- (1) $\{a, b, \{x, y, z\}\} = \{\{a, b, x\}, y, z\} - \{x, \{b, a, y\}, z\} + \{x, y, \{a, b, z\}\},$
- (2) for all $x \in X$ the operators $y \rightarrow \{x, x, y\}$ are Hermitian,
- (3) $\|\{x, x, x\}\| = \|x\|^3$ for all $x \in X$.

A linear mapping $\phi : (X, \{\cdot, \cdot, \cdot\}_X) \rightarrow (Y, \{\cdot, \cdot, \cdot\}_Y)$ between JB^* -triples is a J^* -homomorphism (respectively a J^* -anti-homomorphism) if for $x, y, z \in X$, $\phi(\{x, y, z\}_X) = \{\phi(x), \phi(y), \phi(z)\}_Y$ (respectively $-\{\phi(x), \phi(y), \phi(z)\}_Y$). A deep result of Kaup [K2] states that a complex Banach manifold is biholomorphically equivalent to a bounded symmetric domain if and only if it is biholomorphically equivalent to the open unit ball of a JB^* -triple system (see also [H5]). Moreover, we have three essentially different ways of recognising when two JB^* -triple systems are isomorphic.

¹Cartan factors were discovered by Elie Cartan, while the original Cartan Uniqueness Theorem, mentioned in connection with Theorem 36, is due to his son Henri Cartan.

Proposition 1. *If $(X, \|\cdot\|_X, \{\cdot, \cdot, \cdot\}_X)$ and $(Y, \|\cdot\|_Y, \{\cdot, \cdot, \cdot\}_Y)$ are two JB^* -triple systems, then the following are equivalent.*

1. (Algebraic) *There exists a bijective linear homomorphism $\varphi : X \longrightarrow Y$.*
2. (Geometric) *There exists a surjective linear isometry $\phi : X \longrightarrow Y$, that is $\phi(X) = Y$ and $\|\phi(x)\|_Y = \|x\|_X$ for all $x \in X$.*
3. (Analytic) *There exists a biholomorphic mapping $f : B_X \longrightarrow B_Y$.*

The C^* -algebra $\mathcal{A}$ with triple product $\{a, b, c\} := (ab^*c + cb^*a)/2$ is a JB^* -triple, and hence the open unit ball of a C^* -algebra is a bounded symmetric domain. In particular, $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is a JB^* -triple, and any Banach subspace of $\mathcal{B}(\mathcal{H})$ that is closed with respect to the triple product $\{a, b, c\}$ is called a JC^* -triple. The triple is said to be *associative* if for all $a, b, x, y, z \in X$

$$\{a, b, \{x, y, z\}\} = \{a, \{b, x, y\}, z\} = \{\{a, b, x\}, y, z\}.$$

If l is a positive integer and X is a JB^* -triple with open unit ball B_X , then X and B_X have *rank* l if ℓ_∞^l is isometrically isomorphic to a subspace of X , but this is not the case for any positive integer greater than l . Hilbert spaces are precisely the JB^* -triples of rank 1 and a JB^* -triple X has finite rank if and only if it is reflexive as a Banach space.

If $z, w \in \Delta$, let $\rho(z, w) = \tanh^{-1}(|z - w|/|1 - z\bar{w}|)$. The distance ρ is called the *Poincaré distance* (see [D5]). If $z, w \in \mathcal{D}$, where $\mathcal{D}$ is an arbitrary domain in a Banach space X , let

$$C_{\mathcal{D}}(z, w) = \sup\{\rho(f(z), f(w)) : f \in \mathcal{H}(\mathcal{D}, \Delta)\}$$

and let

$$\delta_{\mathcal{D}}(z, w) = \inf\{\rho(u, v) : f \in \mathcal{H}(\Delta, \mathcal{D}), f(u) = z, f(v) = w\}.$$

We call $C_{\mathcal{D}}$ the *Carathéodory pseudo-distance* on $\mathcal{D}$, and when it separates points, that is, when $C_{\mathcal{D}}(z, w) = 0 \iff z = w$, we call it a distance. The function $\delta_{\mathcal{D}}$ may not satisfy the triangle inequality and so we denote by $K_{\mathcal{D}}$ the largest pseudo-distance smaller than $\delta_{\mathcal{D}}$ and call it the *Kobayashi pseudo-distance* on $\mathcal{D}$. We always have $C_{\mathcal{D}} \leq K_{\mathcal{D}}$, and if $\mathcal{D}$ is a bounded domain then $C_{\mathcal{D}}$ and $K_{\mathcal{D}}$ are both distances. On the unit disc Δ , $\rho = C_{\Delta} = K_{\Delta}$. Holomorphic mappings are contractions and biholomorphic mappings are isometries when the domain and range are both endowed with the Carathéodory (respectively the Kobayashi) distances. The infinitesimal versions of $C_{\mathcal{D}}$ and $K_{\mathcal{D}}$ on the *tangent space* $T(\mathcal{D}) \simeq \mathcal{D} \times X$, $c_{\mathcal{D}}$ and $\kappa_{\mathcal{D}}$, are called (infinitesimal) *pseudo-metrics* and *metrics* as appropriate.

If $\mathcal{D}$ is a bounded domain in $\mathbb{C}^m$ and λ_m is Lebesgue measure on $\mathbb{C}^m$, let

$$H := \{f \in \mathcal{H}(\mathcal{D}) : \|f\|^2 = \int_{\mathcal{D}} |f|^2 d\lambda_m < \infty\}.$$

Then H is a Hilbert space and, if $(\phi_n)_{n=1}^\infty$ is any orthonormal basis for H , let

$$K(z, \bar{z}) = \sum_{n=1}^{\infty} \phi_n(z) \overline{\phi_n(z)}.$$

The function K is the *Bergman kernel function* for $\mathcal{D}$ and we let

$$H_z = \left(\frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K(z, \bar{z}) \right)_{1 \leq j, k \leq m}.$$

If $\gamma : [0, 1] \rightarrow \mathcal{D}$ is piece-wise smooth, we define the length of γ , $l(\gamma)$, by

$$l(\gamma) = \int_0^1 H_{\gamma(t)}(\gamma'(t), \overline{\gamma'(t)})^{1/2} dt,$$

and for $a, b \in \mathcal{D}$ let

$$\rho_{\mathcal{D}}(a, b) = \inf \{ l(\gamma) : \gamma(0) = a, \gamma(1) = b \}.$$

The distance $\rho_{\mathcal{D}}$ is called the *Bergman distance* on $\mathcal{D}$. Since the Bergman distance on Δ coincides with the Poincaré distance, our notation is consistent.

If $f \in \mathcal{H}(\mathcal{D})$, $\mathcal{D}$ a domain in $\mathbb{C}^m$, let

$$(\nabla_z f)(x) = \sum_{j=1}^m \frac{\partial f}{\partial z_j}(z) x_j, \quad \|\nabla_z f\|^2 = \sum_{j=1}^m \left| \frac{\partial f}{\partial z_j}(z) \right|^2$$

where $x = (x_1, \dots, x_m) \in \mathbb{C}^m$ and $z \in \mathcal{D}$, and let

$$Q_f(z) = \sup \{ |(\nabla_z f)x| / H_z(x, \bar{x})^{1/2} : x \neq 0 \in \mathbb{C}^m \}.$$

We refer to the books [C2, C3, D5, D7, E1, L2, M4, M5, P1] and to the fundamental articles [H5, K2, L1] for further details on the above topics.

Bloch functions

Bloch functions of one complex variable and Bloch spaces were motivated by the remarkable 1924 result of Bloch, [B2], mentioned below. Bloch functions of several variables were first defined by Hahn in [H2], but his definition was rather technical and general as it also considered the vector-valued case. Richard, in his 1978 thesis [1], directed by Lee Rubel at the University of Illinois, simplified the definition and confined his investigations, at least initially, to the scalar-valued case. The field is still an active area of research in one, several and infinitely many variables, and most researchers in higher dimensions are guided by Richard's thesis as their initial motivation. In preparing and engaging in this project Richard studied and made a number of contributions, [4, 5, 6, 7, 8], to the one-dimensional theory of Bloch

functions. We discuss these and the necessary background before looking at his thesis. Classical results on Bloch functions may be found in [A1], [C1] and [P2].

If $f \in \mathcal{H}(\mathcal{D})$, $\mathcal{D}$ a domain in $\mathbb{C}$, then $B_r(w)$ is a *schlicht disc* in the range of f if there exists $g \in \mathcal{H}(\Delta, \mathcal{D})$ such that $f \circ g : \Delta \rightarrow B_r(w)$ is bijective. If $f \in \mathcal{H}(\Delta)$, $z \in \Delta$ and $f'(z) \neq 0$, let $d_f(z) := \sup\{r > 0 : B_r(w) \text{ is a schlicht disc in the range of } f\}$. If $f'(z) = 0$ let $d_f(z) = 0$.

In 1970, Pommerenke proved the following result in [P2].

Proposition 2. *If $f \in \mathcal{H}(\Delta)$, then*

$$\sup\{d_f(z) : z \in \Delta\} < \infty \iff \sup_{z \in \Delta}\{(1 - |z|^2)|f'(z)|\} < \infty. \quad (1)$$

Functions that satisfy the equivalent conditions in Proposition 2 are called *Bloch functions*. The original motivating 1925 result of Bloch, [B2], states that there exists a positive constant b , now called the Bloch constant, such that for any non-constant Bloch function $f \in \mathcal{H}(\Delta)$,

$$\sup_{z \in \Delta} d_f(z) \geq b.$$

The precise value of b is unknown. Bloch himself proved that $b > 1/72$, and it is known that $0.43 < b < 0.472$. A Bloch theorem on the open unit ball of a Banach space is given in [H6, theorem 2].

Let $\mathcal{B}(\Delta)$ denote the set of all Bloch functions on Δ , and let

$$\|f\|_{\mathcal{B}(\Delta)} := |f(0)| + \sup_{z \in \Delta}\{(1 - |z|^2)|f'(z)|\}.$$

The space $(\mathcal{B}(\Delta), \|\cdot\|_{\mathcal{B}(\Delta)})$ is a Banach space.

One of Richard's first results was to show in [4] that the unit disc in (1) could be replaced by the set $f^{-1}(E)$, $E \subset f(\Delta)$, as long as the complement of E does not contain large discs. Clearly, $\mathcal{H}^\infty(\Delta) \subset \mathcal{B}(\Delta)$, and in [5] Richard settled several open questions posed by Cima by showing that the universal covering g of the domain $(\overline{\Delta} \cup \mathbb{Z}^2)^c$, where $\mathbb{Z}^2 := \{n + im : n, m \in \mathbb{Z}\}$ denotes the Gaussian integers, is a Bloch function with radial limits of modulus 1 almost everywhere but does not belong to the Hardy spaces $\mathcal{H}^p(\Delta)$ for any p .

In a joint paper with Rubel, [6], Richard proved the following.

Theorem 1. *Let X be a Möbius invariant subspace of $\mathcal{H}(\Delta)$, and let p be a Möbius invariant semi-norm on X . If X admits a non-zero τ_0 -continuous linear functional, then $X \subset \mathcal{B}(\Delta)$ and, for some $A > 0$, $\|f\|_{\mathcal{B}(\Delta)} \leq Ap(f)$ for all $f \in X$.*

As a corollary they show that the only non-zero Möbius invariant closed subspaces

of $\mathcal{H}(\Delta)$ are the whole space and the constant functions. Richard afterwards generalised this result to holomorphic functions on finite-dimensional bounded symmetric domains in [8].

The Bloch functions are one of a family of function spaces parameterised by $[0, 1]$.

Definition 2. For $0 \leq \alpha \leq 1$, let Λ_α denote the set of all $f \in \mathcal{H}(\Delta)$ for which

$$\sup\{|f'(z)|(1 - |z|^2)^{1-\alpha} : z \in \Delta\} < \infty.$$

For $0 < \alpha \leq 1$ this is the usual Lipschitz class while Λ_0 is the Bloch space. Using Taylor series expansions we can identify the space Λ_α with a sequence space, that is we identify the function $z \rightarrow \sum_{n=0}^{\infty} a_n z^n$ with the sequence $(a_n)_{n=0}^{\infty}$. In [7], Richard and his co-authors prove a number of results about the coefficients of functions in Λ_α , among which we find the following theorem.

Theorem 2. For $0 \leq \alpha < 1$ the smallest solid sequence space containing the sequence of coefficients $(a_n)_{n=1}^{\infty}$ of every function $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \Lambda_\alpha$ is the sequence space defined by the condition

$$\sum_{j=n}^{2n} |a_j|^2 = O(n^{-2\alpha}).$$

In fact, given $(a_n)_{n=0}^{\infty}$ satisfying the condition, there exists $\sum_{n=0}^{\infty} a'_n z^n \in \Lambda_\alpha$ with $|a_n| \leq |a'_n|$ for all n .

Richard published the main results on Bloch functions of several complex variables from his thesis in [2] and [3]. The main result in [2] characterises Bloch functions on bounded homogeneous and bounded symmetric domains in $\mathbb{C}^n$ (Theorem 3 below) in an impressive number of ways: (3) is a purely metric condition, (4) and (5) consider a group invariant collection of functions, while a number of the conditions have motivated further research on composition operators. Richard's definition of a *Bloch function* on an arbitrary bounded domain $\mathcal{D} \subset \mathbb{C}^m$ is condition 1 in Theorem 3.

Theorem 3. Let $\mathcal{D}$ denote a bounded homogeneous domain in $\mathbb{C}^m$. For $f \in \mathcal{H}(\mathcal{D})$ the following conditions are equivalent.

- (1) $\sup\{Q_f(z) : z \in \mathcal{D}\} < \infty$.
- (2) The radii of the schlicht discs in the range of f are bounded above.
- (3) $f : (\mathcal{D}, \rho_{\mathcal{D}}) \rightarrow (\mathbb{C}, |\cdot|)$ is uniformly continuous.
- (4) The family

$$\{(f \circ \phi)(z) - f(\phi(z_0)) : \phi \in \text{Aut}(\mathcal{D})\}$$

is a normal family for some (every) $z_0 \in \mathcal{D}$.

(5) The supremum

$$\sup \{ \|\nabla_{z_0}(f \circ \phi)\| : \phi \in \text{Aut}(\mathcal{D}) \}$$

is finite for some (every) $z_0 \in \mathcal{D}$.

(6) The family $\{f \circ g : g \in \mathcal{H}(\Delta, \mathcal{D})\}$ is a family of Bloch functions with uniformly bounded Bloch norm.

(7) The family $\{z \longrightarrow (f \circ g(z) - f(g(0))) : g \in \mathcal{H}(\Delta, \mathcal{D})\}$ is a normal family.

If, in addition, $\mathcal{D}$ is a bounded symmetric domain the above are equivalent to:

(8) The absolute convex hull of the orbit of f under $\text{Aut}(\mathcal{D})$ is a normal family.

For specific domains, such as the m -dimensional polydisc and the complex Euclidean ball in $\mathbb{C}^m$, B_m , Richard obtained more explicit results. From now on we only consider domains that contain the origin.

Theorem 4. *The following are equivalent for $f \in \mathcal{H}(B_m)$.*

(1) f is a Bloch function.

(2) $\|f\|_m := |f(0)| + \sup_{z \in B_m} \|\nabla_z f\|(1 - \|z\|^2) < \infty$.

(3) For some $K > 0$,

a. if $z \in B_m$,

$$|(\nabla_z f)(z)|(1 - \|z\|^2) \leq K\|z\|,$$

b. if $z \in B_m, x \in \mathbb{C}^m, \langle x, \bar{z} \rangle = 0$,

$$|(\nabla_z f)(x)|(1 - \|z\|^2)^{1/2} \leq K.$$

(4) If $f_x(z) = f(zx)$ for $x \in \mathbb{C}^m, \|x\| = 1$, and $z \in \Delta$, then $f_x \in \mathcal{B}(\Delta)$ and

$$\sup_{\|x\|=1} \|f_x\|_{\mathcal{B}(\Delta)} < \infty.$$

Note that the above shows the following unexpected result: if $f \in \mathcal{H}(B_m)$ and $\|\nabla f\|$ grows at most as fast as $1/(1 - \|z\|^2)$, then the directional derivatives of f in the directions perpendicular to the radial directions grow more slowly. In fact, they grow at most as fast as $1/(1 - \|z\|^2)^{1/2}$.

If $f \in \mathcal{B}(\mathcal{D})$, the set of Bloch functions on $\mathcal{D}$, let

$$\|f\|_{\mathcal{B}(\mathcal{D})} = |f(0)| + \sup\{Q_f(z) : z \in \mathcal{D}\}.$$

Richard proved the following result in [4] and proved Theorem 6 in [5].

Theorem 5. *If $\mathcal{D}$ is a bounded homogeneous domain in $\mathbb{C}^m$, then*

$$(\mathcal{B}(\mathcal{D}), \|\cdot\|_{\mathcal{B}(\mathcal{D})})$$

is a Banach space and $\|\cdot\|_{\mathcal{B}(B_m)}$ and $\|\cdot\|_m$ are equivalent norms on $\mathcal{B}(B_m)$.

Theorem 6. *If m is a positive integer, then*

$$\mathcal{B}_0(B_m) := \{f \in \mathcal{B}(B_m) : \lim_{\|z\| \rightarrow 1} \|\nabla_z f\|(1 - \|z\|^2) = 0\} \cong c_0$$

and

$$(\mathcal{B}_0(B_m), \|\cdot\|_{\mathcal{B}(B_m)})'' \cong (\mathcal{B}(B_m), \|\cdot\|_{\mathcal{B}(B_m)}) \cong \ell_\infty.$$

This gives a complete characterisation of the Banach space structure for the Bloch space on finite-dimensional bounded symmetric domains of rank 1. In [5], Richard considered the same questions for spaces of rank greater than 1. He found that the situation was much more complicated and that the analogue of Theorem 5 does not hold. He did show, however, using topological tensor products, that the Bloch functions on finite-dimensional polydiscs are dual Banach spaces.

After publishing the results in his thesis, Richard wrote two further papers on Bloch functions. The main result in [8] is an extension to bounded symmetric domains of Theorem 1.

Theorem 7. *Let $\mathcal{D}$ be a finite-dimensional irreducible bounded symmetric domain. If $L \neq 0$ is a continuous linear functional on $\mathcal{H}(\mathcal{D})$ and $f \in \mathcal{H}(\mathcal{D})$ satisfies*

$$\sup\{|L(f \circ \phi)| : \phi \in \text{Aut}(\mathcal{D})\} = M < \infty,$$

then $f \in \mathcal{B}(\mathcal{D})$ and $\|f\|_{\mathcal{B}(\mathcal{D})} \leq c_L M$ where c_L depends only on L .

For non-irreducible domains, however, the space of Bloch functions had to be replaced by a function space defined by partial derivatives of order r , where r is the number of irreducible domains in the decomposition. The proofs involved in both Theorem 7 and its non-irreducible generalisation are very technical, decidedly non-trivial and involve preliminary results on a maximal strongly orthogonal set of non-compact positive roots of the associated Lie algebras. We do not go into details.

Motivated by results in his thesis, Richard introduced in [9] the concept of a *natural function space* as a semi-normed subspace (X, p) of $\mathcal{H}(\Delta)$ such that $\{f \in X : p(f) \leq 1\}$ is a closed subset of $(\mathcal{H}(\Delta), \tau_0)$. He proved that the space of Bloch functions is the largest Möbius invariant function space on Δ that is also a natural function space, and proved, with fewer assumptions, a number of known results from [A3] and [A4], including a characterisation of the Dirichlet space.

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Infinite-dimensional complex analysis

In the early 1980s, Richard and I began investigating bounded symmetric domains in infinite-dimensional Banach spaces. Neither of us had previously worked in this area, nor was there anyone else in Dublin at that time interested in the topic, but each of us had experienced a different part of the required background for such a study. Our research eventually extended to eleven joint papers, some with additional authors. Although different research areas may appear to have little in common, and even though the number of mathematical ideas and techniques is very large, we often see, if areas are close to one another, similar ideas and concepts appearing in both, perhaps with a different emphasis and perhaps in semi-disguise. This helps in moving between research areas. For instance, in examining a bounded symmetric domain $\mathcal{D}$ in a Banach space X , an important role is played by the group of biholomorphic automorphisms of $\mathcal{D}$, $\text{Aut}(\mathcal{D})$, while the corresponding linear collection, the set of linear isometries of X , is not as crucial in the linear theory of Banach spaces. Looking back now, years later, it is interesting to observe two themes that held our attention as we made the transition from a linear to a non-linear area. The construction of metrics (Caratheodory, Kobayashi, Bergman) adapted to the given mappings of interest (holomorphic) is normally the reverse of the approach taken in functional analysis, which looks at the functions (linear) that are adapted to the given norm (metric). Cartan's classification of finite-dimensional bounded symmetric domains, with its decomposition into a finite product of irreducible factors and the concrete list of irreducible (bounded symmetric) domains, is unlike any result in Banach space theory.

For the following result, proved in [14], X is a Banach space, and a domain is irreducible if it is not biholomorphically equivalent to a product of two domains. Let $\phi_X(r)$ denote the maximum number of irreducible factors for a domain $\mathcal{D}$ in X when $B_X \subset \mathcal{D} \subset rB_X$.

Theorem 8. (1) *Each bounded domain in X is biholomorphically equivalent to a finite product of irreducible domains if and only if $c_0 \not\subset X$.*
 (2) *$\phi_X(r) < \infty$ for all $r \geq 1$ if and only if X has finite cotype.*
 (3) *If $q(X) := \inf\{q : X \text{ has cotype } q\} < \infty$, then for every $\varepsilon > 0$ there is*

$A(\varepsilon, X) > 0$ such that

$$\left(\frac{r-1}{2}\right)^{q(X)} \leq \phi_X(r) \leq A(\varepsilon, X)r^{q(X)+\varepsilon}.$$

- (4) If $2 \leq p < \infty$, $X = \mathcal{L}^p(\Omega, \mu)$ is infinite-dimensional and $[\cdot]$ denotes integer part, then $\phi_X(r) = [r^p]$.

Part (4) of Theorem 8 for $p = 2$ is due to Kaup [K1]. Note that if X is an infinite-dimensional Hilbert space and $\mathcal{D}$ is a domain in X , $B_X \subset \mathcal{D} \subset \sqrt{2}B_X$, then $\mathcal{D}$ is irreducible.

When domains cannot be written as a product of a finite number of irreducible domains, one has to look at infinite products. These come in various forms. After finite products the simplest case is a countably infinite product, that is, the indexing set is discrete and countable. As infinite products of Banach spaces are not Banach spaces, however, some additional structure is required. The following result, influenced by Cartan's decomposition result, is proved in [12].

Theorem 9. *Let $\{X_n\}_{n \in \mathbb{N}}$ denote a 1-unconditional finite-dimensional decomposition of the Banach space X , and suppose the open unit ball of X is a bounded symmetric domain. Then X is isometrically isomorphic to $(\sum_m F_m)_{c_0(I)}$, I finite or countably infinite, and the unit ball of F_m is either an irreducible finite-dimensional bounded symmetric domain or else F_m is isometrically isomorphic to $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$, where $\mathcal{H}_1$ is an infinite-dimensional separable Hilbert space and $\mathcal{H}_2$ is a finite-dimensional Hilbert space.*

We now describe a more general situation where the indexing set is a topological space and refer to [B1], especially chapter 4, for further details. If X is a Banach space, then a continuous linear operator $T : X \rightarrow X$ is a *multiplier* if each extreme point ϕ of B_{X^*} is an eigenvector of tT with eigenvalue $\lambda_T(\phi)$. The *centraliser* of X , $Z(X)$, consists of those multipliers for which there is another multiplier S such that $\lambda_S(\phi) = \overline{\lambda_T(\phi)}$. The centraliser of a Banach space with its norm and algebraic structure inherited from $\mathcal{B}(X)$ is a commutative $\mathcal{C}^*$ -algebra, and hence is isometrically isomorphic to $(\mathcal{C}(K), \|\cdot\|_K)$ for some compact Hausdorff space K . This is the same K that appears in Definition 3 below.

Definition 3. *A maximal function module representation of the Banach space X is a quadruple $(K, (X_k)_{k \in K}, \tilde{X}, \varrho)$ where K is a compact Hausdorff space, $(X_k)_k$ is a collection of Banach spaces, $\tilde{X}$ is a Banach subspace of $\ell^\infty(\{X_k\}_{k \in K})$ and $\varrho : X \rightarrow \tilde{X}$, $\varrho(x) = (x_k)_{k \in K}$, is an isometric isomorphism such that*

- (1) $X_k = \{x_k : x \in X\}$ for $k \in K$,
- (2) $\{k : X_k \neq 0\}$ is a dense subset of K ,
- (3) for each $x \in X$ the mapping $k \in K \rightarrow \|x_k\|$ is upper semi-continuous,

- (4) X is a $\mathcal{C}(K)$ module, that is, if $h \in \mathcal{C}(K)$, $x \in X$ and $\varrho(x) = (x_k)_{k \in K} \in \tilde{X}$, then $hx := (h(k)x_k)_{k \in K} \in \tilde{X}$ and $\|hx\| \leq \|h\|_K \cdot \|x\|$,
 (5) $Z(X) = \{\varrho^{-1} \circ m_h \circ \varrho : h \in \mathcal{C}(K)\}$ where

$$m_h : \ell^\infty(\{X_k\}_{k \in K}) \longrightarrow \ell^\infty(\{X_k\}_{k \in K})$$

is given by $m_h[(x(k))_{k \in K}] = (h(k)x(k))_{k \in K}$.

Every Banach space has a maximal function module representation. This is unique in a canonical way and, moreover, a Banach space X is irreducible if and only if K is connected (a Banach space X is defined to be irreducible if it cannot be written isometrically as $X = Y \oplus_\infty Z$ where the Banach spaces Y and Z are non-trivial). In particular, the Banach space $(\mathcal{C}(K), \|\cdot\|_K)$ is irreducible if and only if K is connected. If $Z(X)$ is one-dimensional, or equivalently if K is a singleton, then X is irreducible. If X is a dual Banach space, then K is *extremely disconnected*, that is the closure of every open subset of K is open, and the mappings in condition (3) above are continuous. Moreover, a dual Banach space is irreducible if and only if $Z(X)$ is one-dimensional.

For our next result we require the *Radon-Nikodým property* (RNP). Many different geometric, analytic and measure theoretic characterisations of this property are known (see, for instance, [D2] or [D7, p. 111]). For example, X has (RNP) if and only if every function of bounded variation $f : [0, 1] \rightarrow X$ is differentiable almost everywhere with respect to Lebesgue measure on $[0, 1]$. Later, we will need a more general property, the *analytic Radon-Nikodým property* (aRNP). A Banach space X is said to have aRNP if each $f \in \mathcal{H}(\Delta, B_X)$ has radial limits almost everywhere. Our next result, Theorem 10, is proved in [16]. The space X is a Banach space and the notation from Definition 3 is retained.

- Theorem 10.** (1) If X has RNP, then $X' = \ell^\infty(\{Y_i\}_{i \in I})$ where each Y_i is an irreducible dual Banach space.
 (2) If X has a 1-unconditional finite-dimensional decomposition, then X is isometrically isomorphic to $(\sum_m X_m)_{c_0(I)}$ where I is countable, each X_m is irreducible, and $Z(X_m'')$ is one-dimensional.
 (3) If ψ in $\text{Aut}(B_X)$, then there exists a homeomorphism δ of K , and for each $k \in K$, $\theta_k \in \text{Aut}(B(X_k))$ and an isometry $w_k : X_k \rightarrow X_{\delta(k)}$, such that for all $x \in B_X$ and all $k \in K$,

$$\varrho(\psi(x))(\delta(k)) = w_k(\theta_k(\varrho(x)(k))).$$

- (4) The space X may be isometrically embedded as a weak*-dense subspace of $Y := \ell^\infty(\{Y_i\}_{i \in I})$ where each Y_i is an irreducible dual space such that each $\psi \in \text{Aut}(B_X)$ extends to an element of $\text{Aut}(B_Y)$.

The *centroid* $C(X)$ of a JB^* -triple $(X, \{\cdot, \cdot, \cdot\})$ consists of all $T \in \mathcal{B}(X)$ such

that $T(\{x, y, z\}) = \{Tx, y, z\}$ for all $x, y, z \in X$. This algebraic structure leads to an alternative approach in [17] and to the following result.

Theorem 11. *If X is a JB^* -triple system then $C(X) = Z(X)$.*

A compact Hausdorff space K together with a continuous mapping $\sigma : \mathbb{T} \times K \longrightarrow K$, $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, such that $\sigma(\alpha, \sigma(\beta, x)) = \sigma(\alpha\beta, x)$ and $\sigma(1, x) = x$ for all $x \in K$ and $\alpha, \beta \in \mathbb{T}$, is called a T_σ -space. If (K, σ) is a T_σ space, then the Banach space X is called a C_σ -space if it is isometrically isomorphic to

$$C_\sigma(K) := \{f \in \mathcal{C}(K) : f(\sigma(\alpha, x)) = \alpha f(x), (\sigma, x) \in \mathbb{T} \times K\}.$$

An application of Theorem 11 yields, in [17], the following characterisation of associative JB^* -triples (see also [D5] and [H5]).

Theorem 12. *For a JB^* -triple $(X, \{\cdot, \cdot, \cdot\})$, the following are equivalent:*

- (1) *X is associative (also called Abelian or commutative),*
- (2) *$\{x, y, \{z, u, v\}\} = \{\{x, y, z\}u, v\}$ for all (x, y, z, u, v) in X ,*
- (3) *X is a C_σ -space,*
- (4) *each X_k in the maximal function module representation of X has dimension at most one,*
- (5) *all extreme points of $B_{X'}$ are either J^* -homomorphisms or J^* -anti-homomorphisms,*
- (6) *$Z(X) = \{T_1 + iT_2 : T_1, T_2 \text{ Hermitian on } X\}$.*

We next consider (holomorphic) metrics and distances on domains in Banach spaces. A metric $\alpha : B_X \times X \longrightarrow \mathbb{R}$ on the tangent space $T(B_X) \simeq B_X \times X$, where B_X is the unit ball of the JB^* -triple X , is a $\mathcal{C}^2$ -invariant Hermitian metric if (a) α^2 is a $\mathcal{C}^2$ mapping, (b) α restricted to $\{x\} \times X$ is a Hilbert norm for each $x \in B_X$ and (c) $\alpha(x, v) = \alpha(\varphi(x), \varphi'(x)v)$ for all $(x, v) \in B_X \times X$ and all $\varphi \in \text{Aut}(B_X)$. The Bergman metric on finite-dimensional bounded domains is such a metric, but due to the non-existence of a translation invariant Lebesgue measure on any infinite-dimensional locally convex space, the Bergman metric cannot be defined on arbitrary infinite-dimensional domains. A JB^* -triple X admits an invariant Hermitian metric if and only if it is of finite rank, that is, if and only if it is a reflexive Banach space or, equivalently, isomorphic to a Hilbert space.

Using Möbius transformation, tripotents, complex extreme points, the Pierce decomposition and invariant inner products, the metrics h_X and h_Y in the following theorem are explicitly calculated and give the best possible constant in the Schwarz-Pick lemma. We call $h_{(\cdot)}$ the algebraic metric of Harris. The Bergman metric is a multiple of h_X when B_X is a finite-dimensional irreducible bounded symmetric domain. Irreducibility is neither necessary nor sufficient in general, however. Theorem 13 is proved in [19] (see also [D5] and [H7]).

Theorem 13. *If X and Y denote JB^* -triples of finite rank, then there exist invariant Hermitian metrics, h_X on B_X and h_Y on B_Y , such that:*

- (1) (Schwarz-Pick lemma) *If $f \in \mathcal{H}(B_X, B_Y)$, then for all $(x, v) \in X_0 \times X$,*

$$h_Y(f(x), f'(x)v) \leq \sqrt{r}h_X(x, v)$$

whenever Y has rank r .

- (2) *If X has rank r , α is an invariant Hermitian metric on B_X , M is a positive constant, and each $f \in \mathcal{H}(B_X, B_X)$ satisfies $\alpha(f(x), f'(x)v) \leq M\alpha(x, v)$ for all $(x, v) \in B_X \times X$, then $M^2 \geq r$. If $M^2 = r$, then α is a positive multiple of h_X .*

The use of invariant inner-products in [19] motivated the approach in [22]. An *invariant inner-product* on a Banach space X is an inner-product $\langle, \rangle$ that generates the topology of X and is invariant under the isometries of X . Haar measure shows that every finite-dimensional Banach space admits an invariant inner-product. If the Banach space X admits an invariant inner-product, that is, a scalar multiple of any other invariant inner-product on X , then we say that X has the *unique invariant inner-product property*. If X has a unit vector basis $(e_j)_{j=1}^n$ and norm satisfying $\|\sum_{j=1}^m a_j e_j\| = \|\sum_{j=1}^m \lambda_j a_j e_{\pi(j)}\|$ for any scalars $\lambda_j \in \mathbb{T}$ and any permutation π of $\{1, 2, \dots, m\}$, then X has the unique invariant inner-product property, and any invariant inner-product on X is a scalar multiple of

$$\left\langle \sum_{j=1}^m a_j e_j, \sum_{j=1}^m b_j e_j \right\rangle := \sum_{j=1}^m a_j \overline{b_j}.$$

For X isomorphic to a Hilbert space H we let $e(X)$ denote the *Banach-Mazur distance* from X to H , that is,

$$e(X) = \inf\{\|T\| \cdot \|T^{-1}\| : T \in \mathcal{L}(X, H) \text{ is invertible}\}.$$

If X is a JB^* -triple of rank r , then $e(X) = \sqrt{r}$. If $(X, \|\cdot\|)$ has the unique invariant inner-product property, and $\langle, \rangle$ is any invariant inner-product on X , then (see [19]),

$$e(X)^2 = \frac{\sup_{\|x\|=1} \langle x, x \rangle}{\inf_{\|x\|=1} \langle x, x \rangle}.$$

We confine ourselves to presenting a finite-dimensional Schwarz lemma from [22].

Theorem 14. *Let $X = (X_1 \oplus X_2 \dots \oplus X_m)_{\ell_p}$, $2 \leq p \leq \infty$, where each X_j has finite dimension d_j and the unique invariant inner product property. Let $1/r + 2/p = 1$ and $d = \sum_{j=1}^m d_j$. If H is a d -dimensional Hilbert space, then*

$$\max\{|\det(f'(0))| : f \in \mathcal{H}(B_X, B_H), f(0) = 0\} = d^{-d/2r} \prod_{j=1}^m \left(\frac{d_j^{1/r}}{e(X_j)^2} \right)^{d_j/2}.$$

In [L1], Lempert proved that $C_{\mathcal{D}} = K_{\mathcal{D}} = \delta_{\mathcal{D}}$ whenever $\mathcal{D}$ is a convex domain in $\mathbb{C}^m$, and in [10] this result is extended, using Montel's theorem and ultrafilters, to arbitrary infinite-dimensional spaces. A *complex geodesic* in a domain $\mathcal{D}$ in a Banach space X is a mapping $\phi \in \mathcal{H}(\Delta, \mathcal{D})$ such that $\rho(u, v) = C_{\mathcal{D}}(\phi(u), \phi(v))$ for two distinct points (and hence all pairs of points) in Δ . In such a case, we say that u and v are joined by the complex geodesic ϕ . Existence, uniqueness, extensions to the boundary, explicit formulae and applications are the main questions regarding complex geodesics. Vesentini [V1] proved that the origin and a non-zero $x \in B_X$ are joined by a geodesic that is unique modulo reparametrisation if and only if $x/\|x\|$ is a complex extreme point of the closed unit ball in X . The Banach space X is *strictly convex* if every unit vector x is a real extreme point of the closed unit ball, that is, if $x = \lambda y + (1 - \lambda)z$ where $\|y\| \leq 1$ and $\|z\| \leq 1$, then $x = y = z$. If $(X, \|\cdot\|)$ is a Banach space, let

$$\delta_{B_X}(z, v) := \sup\{r > 0 : z + rv\Delta \subset B_X\}$$

when $\|z\| < 1$ and $\|v\| = 1$. If $\epsilon > 0$, let

$$\delta_X(\epsilon) = \sup\{\delta_{B_X}(z, v) : \|z\| \leq 1 - \epsilon, \|v\| = 1\}.$$

The Banach space X is *complex uniformly convex* if $\delta_X(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$. For example, the predual of a von Neumann algebra is complex uniformly convex. The following theorem is proved in [21] (see also [D6]). Part (a), when X is a dual Banach space, is given in [10], while part (c) falls short of being a converse to a result in [G2], which states that if all complex geodesics have continuous extensions to the boundary, then all unit vectors are complex extreme points of the closed unit ball.

Theorem 15. (a) *If the Banach space X is 1-complemented in its bidual, then any two points in B_X can be joined by a complex geodesic.*

(b) *If the Banach space X has aRNP, and B_X is strictly convex, then there exists, up to reparametrisation, at most one geodesic joining any two points in B_X .*

(c) *If $(X, \|\cdot\|)$ is a complex uniformly convex Banach space, and there exist $A > 0$ and $s > 0$ such that $\delta_X(\epsilon) \leq A\epsilon^s$, then any complex geodesic in B_X extends continuously to $\overline{\Delta}$.*

(d) *Each complex geodesic in B_{ℓ_p} , $1 \leq p < \infty$, has, after reparametrisation, the form $\phi = (\phi_j)_{j=1}^\infty$, $\phi_j \in \mathcal{H}(\Delta)$ for each j ,*

$$\phi_j(z) = c_j \left(\frac{z - \alpha_j}{1 - \overline{\alpha_j}z} \right)^{\beta_j} \left(\frac{1 - \overline{\alpha_j}z}{1 - \overline{\gamma}z} \right)^{2/p}$$

where $\gamma \in \Delta$, $\alpha_j \in \overline{\Delta}$, $c_j \in \mathbb{C}$, $\beta_j = 0$ or 1 , and

$$\sum_{j=1}^{\infty} |c_j|^p (1 + |\alpha_j|^2) = 1 + |\gamma|^2, \quad \sum_{j=1}^{\infty} |c_j|^p \alpha_j = \gamma.$$

To present structural results from [13] on JB^* -triples, we need a number of preliminary definitions (for more general details we refer to [B1] and [H4]). A closed subspace Y in a Banach space X is an L -summand (respectively an M -summand) if Y is complemented in X and there exists a non-zero complement Z such that $X = Y \oplus_1 Z$ (respectively $X = Y \oplus_{\infty} Z$). A closed subspace Y in a Banach space X is an M -ideal if $Y^{\perp} := \{\varphi \in X', \varphi(Y) = 0\}$ is an L -summand in X' . A JB^* -triple that is also isometrically isomorphic to a dual Banach space is called a JBW^* -triple.

A JBW^* -triple X is called a *triple factor* if it is irreducible, that is, if it is not isometric to an ℓ_{∞} -sum of JB^* -triples. A closed subspace J of a JB^* -triple, $(X, \{\cdot, \cdot, \cdot\})$, is an *ideal* if $\{a, u, v\} \in J$ whenever $a \in J$ and $u, v \in X$ (see [19]). A *triple factor representation* of a JB^* -triple $(X, \{\cdot, \cdot, \cdot\}_X)$ is a linear homomorphism φ preserving the triple product from X onto a w^* -dense subspace of a triple factor $(Y, \{\cdot, \cdot, \cdot\}_Y)$. A collection of representations $(\phi_i)_{i \in I}$ is *faithful* if $\bigcap_{i \in I} \ker(\phi_i) = \{0\}$.

Theorem 16. *Let X be a JB^* -triple.*

- (1) *If X is a JBW^* -triple and a dual Banach space, then X has a unique predual X_* , and the triple product on X is separately $\sigma(X, X_*)$ -continuous.*
- (2) *The closed ideals in X are precisely the M -ideals.*
- (3) *X has a faithful family of triple factor representations.*

Our final results in this section are taken from [18] and [20] and were motivated by entirely different considerations to the above – for background information we refer to [D1] and [D7]. A Schauder basis $(e_n)_{n=1}^{\infty}$ in a locally convex space X is an *absolute basis* if for each continuous semi-norm p on X there exists a continuous semi-norm q such that

$$\sum_{n=1}^{\infty} p(z_n e_n) \leq q\left(\sum_{n=1}^{\infty} z_n e_n\right)$$

for all $\sum_{n=1}^{\infty} z_n e_n \in X$. The *Grothendieck-Pietsch criterion* for nuclearity says that a locally convex space X with Schauder basis $(e_n)_{n=1}^{\infty}$ is nuclear if and only if for each continuous semi-norm p on X there exists a continuous semi-norm q on X such that $\sum_{n=1}^{\infty} p(e_n)/q(e_n) < \infty$ (the convention $0/0 = 0$ is used if required). If X is a locally convex space with basis $(e_n)_{n=1}^{\infty}$, then the monomials are the finite products of coordinate functionals derived from the Schauder basis. Let $\mathbb{N}_0$ denote the non-negative integers and let $\mathbb{N}_0^{(\mathbb{N})}$ denote the set of all finite sequences with entries from $\mathbb{N}_0$. If $m = (m_1, \dots, m_n) \in \mathbb{N}_0^{(\mathbb{N})}$, then, letting $z^0 = 1$ all z , we denote

by z^m the monomial mapping

$$\sum_{n=1}^{\infty} z_n e_n \in X \longrightarrow z_1^{m_1} \cdots z_n^{m_n} =: z^m.$$

If $P := \{\sum_{n=1}^{\infty} z_n e_n : |z_n| \leq \alpha_n\}$ is a polydisc in X , then $\|z^m\|_P = \alpha^m$. If X is a Fréchet nuclear (or a *DFN*) space with basis, then the monomials are an absolute basis for $(\mathcal{H}(X), \tau_0)$. Since the compact polydiscs are a fundamental sequence of compact subsets of X , this is equivalent to the following: for each compact polydisc $P := \{\sum_{n=1}^{\infty} z_n e_n : |z_n| \leq \alpha_n\}$ in X , there exists $C > 0$ and a compact polydisc $P' := \{\sum_{n=1}^{\infty} z_n e_n : |z_n| \leq \beta_n\}$ in X such that

$$\sum_{m \in \mathbb{N}_0^{(\mathbb{N})}} |a_m| \alpha^m = \sum_{m \in \mathbb{N}_0^{(\mathbb{N})}} |a_m| \cdot \|z^m\|_P \leq C \left\| \sum_{m \in \mathbb{N}_0^{(\mathbb{N})}} a_m z^m \right\|_{\{|z_n| \leq \beta_n, n=1,2,\dots\}} \quad (2)$$

for all $\sum_{m \in \mathbb{N}_0^{(\mathbb{N})}} a_m z^m \in \mathcal{H}(X)$. Motivated by linear results on the relationship between absolute convergence and nuclearity we considered the following question. If the locally convex space X with basis contains a fundamental system of compact sets consisting of polydiscs, and the monomials form an absolute basis for $(\mathcal{H}(X), \tau_0)$, is X' endowed with the compact open topology nuclear?

Since (2) can be replaced by an infinite number of inequalities, each involving only a finite number of variables, a re-scaling suggests that information about sequences $(\gamma_n)_{n=1}^{\infty}$ for which there exists $C > 0$ such that

$$\sum_{m \in J} |a_m| \gamma^m \leq C \left\| \sum_{m \in J} a_m z^m \right\|_{\Delta^n}$$

for all finite subsets J of $\mathbb{N}_0^{(\mathbb{N})}$ might lead, via the Grothendieck-Pietsch criterion, to nuclearity. A one-dimensional result of Bohr from 1914 states that if $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathcal{H}^{\infty}(\Delta)$ and

$$\sum_{n=0}^{\infty} |a_n| r^n \leq \left\| \sum_{n=0}^{\infty} a_n z^n \right\|_{\Delta} \quad (3)$$

then $r \leq 1/3$ and this is best possible (see [B6]). This implies that $\gamma_n \leq 1/3$ for all n , and suggests that a suitable generalisation of Bohr's result would give us the required result. It was an estimate on k -fold symmetric random tensors in [M1], however, that allowed us in [18] to prove the holomorphic result (Theorem 17) and at the same time to generalise Bohr's inequality (Theorem 18).

Theorem 17. *If X is a Montel space with basis, and the monomials form an absolute basis for $(\mathcal{H}(X), \tau_0)$, then (X', τ_0) is nuclear.*

Theorem 18. *If $(r_i)_{i=1}^\infty$ is a sequence of non-negative real numbers and*

$$\sum_{m \in \mathbb{N}_0^{(\mathbb{N})}} |a_m| r^m \leq \left\| \sum_{m \in \mathbb{N}_0^{(\mathbb{N})}} a_m z^m \right\|_{\{|z_i| < 1, i \in \mathbb{N}\}}$$

for any collection of complex numbers $(a_m)_{m \in \mathbb{N}_0^{(\mathbb{N})}}$, then for each $\epsilon > 0$ there exists $C(\epsilon) > 0$ such that for all n , $\sum_{j=1}^n r_j \leq C(\epsilon) n^{\frac{1}{2} + \epsilon}$.

Bohr was interested in Dirichlet series of one complex variable and wished to calculate the supremum S of the difference in width between the half-planes of uniform and absolute convergence. He showed that $S \geq 2$, and Toeplitz showed that $S \leq 4$. Using our methods from [18] we proved in [20] that $S = 2$, unaware that Bohnenblust and Hille had previously proved this result in 1932 (see [B5]).

In researching one-dimensional Dirichlet series, Bohr, using the well-known decomposition of natural numbers into the product of primes, discovered an intimate connection between one-dimensional Dirichlet series and power series in an infinite number of variables (see [D7, p. 231]). Theorem 18 attracted interest in proving a several-variables Bohr inequality, see for instance [B3] and [B4], but it was Defant and his collaborators who understood the significance of Bohr's general approach and who unlocked the potential hidden in inequality (3) and in the estimates presented in [B5]. After over twenty years of focused effort, we were finally presented in 2019 with the 680 page monograph [D1] by Defant, Garcia, Maestre and Sevilla-Peris. This monograph has three interconnected topics: Dirichlet series, infinite-dimensional holomorphy and harmonic analysis. Moreover, its methods and results touch many other topics, such as approximation theory, random polynomials, several complex variables theory, tensor products, probability theory, Hardy spaces, number theory and geometric properties of Banach spaces, and portrays a fascinating gallery of named inequalities. Overall, it is a powerful witness to the unity of mathematics.

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Elementary operators

The simplest of all operators from an algebra A to itself are the left and right multiplication by elements of A , $L_a : x \rightarrow ax$ and $R_a : x \rightarrow xa$. Moreover, the axiom for a Banach algebra, $\|xy\| \leq \|x\| \cdot \|y\|$, ensures that $\|L_a\| \leq \|a\|$ and $\|R_a\| \leq \|a\|$. Operators in the subalgebra of $\mathcal{B}(A)$ generated by L_a and R_a , where a ranges over A , are called *elementary operators*. Hence, an elementary operator $A \rightarrow A$ is one that may be presented in the form

$$T_{\mathbf{a}, \mathbf{b}}(x) = \sum_{i=1}^l a_i x b_i,$$

where a_i and b_i ($1 \leq i \leq l$) belong to A , $\mathbf{a} = (a_1, \dots, a_l)$ and $\mathbf{b} = (b_1, \dots, b_l)$. One can consider when A is non-unital, as Richard does in certain situations, the multiplier algebra of A (see [M5, p. 39]), but we refrain from going into that level of generality here. Inner derivations, that is, mappings of the form $\delta_a : x \rightarrow ax - xa$, where $a \in A$, are elementary operators.

The above representation of $T_{\mathbf{a}, \mathbf{b}}$ may not be unique, that is, there may exist k -tuples $\mathbf{a}' = (a'_1, \dots, a'_k)$ and $\mathbf{b}' = (b'_1, \dots, b'_k)$ such that $T_{\mathbf{a}, \mathbf{b}} = T_{\mathbf{a}', \mathbf{b}'}$. The smallest l in a representation of $T_{\mathbf{a}, \mathbf{b}}$ is called the *minimal length* of $T_{\mathbf{a}, \mathbf{b}}$. If $A = \mathcal{B}(\mathcal{H})$ and both $\{a_1, \dots, a_l\}$ and $\{b_1, \dots, b_l\}$ consist of linearly independent sets, then $T_{\mathbf{a}, \mathbf{b}}$ has minimal length l .

If A is an algebra we let $M_k(A)$ denote the set of all $k \times k$ matrices with entries in A . If $\mathcal{H}$ is a Hilbert space and k is a positive integer we have a natural identification $M_k(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\ell_2^k(\mathcal{H}))$. Hence, if A is a C^* -algebra and $T \in \mathcal{B}(A)$, we may use an embedding of A in $\mathcal{B}(\mathcal{H})$ and the above identification to define $T^{(k)} : M_k(A) \rightarrow M_k(A)$ by letting

$$T^{(k)}([x_{i,j}]_{1 \leq i,j \leq k}) = [T(x_{i,j})]_{1 \leq i,j \leq k}.$$

If $\sup_k \|T^{(k)}\| < \infty$, then T is said to be *completely bounded* and we let $\|T\|_{cb} = \sup_k \|T^{(k)}\|$. Elementary operators are completely bounded, and the Haagerup norm, as given in the following formula, coincides with the completely bounded norm when

$A = \mathcal{B}(\mathcal{H})$:

$$\|T_{\mathbf{a}, \mathbf{b}}\|_{cb} = \inf \left(\left\| \sum_{j=1}^l a_j a_j^* \right\| \cdot \left\| \sum_{k=1}^l b_k^* b_k \right\| \right)^{1/2}$$

where the infimum is taken over all representations of $T_{\mathbf{a}, \mathbf{b}}$.

If $T^{(k)}$ is positive we say that T is k -positive, and if T is k -positive for all k we say that T is *completely positive*. The standard embedding of $M_k(\mathcal{H})$ in $M_{k+1}(\mathcal{H})$ shows that $(k+1)$ -positivity implies k -positivity.

The experience in his thesis with the unit ball of $M_n(\mathbb{C})$ as a domain for Bloch functions, and afterwards as a prototype JB^* -triple, sparked Richard's interest in operator spaces as the subject developed, although almost twenty years would pass before he began his own researches on operator spaces. In 1997 he typed up a short survey on Pisier's operator Hilbert space and gave seminars on the subject, showing, in particular, that the row and column Hilbert operator spaces were not isomorphic as operator spaces. It is thus not surprising that his first publication on elementary operators, [24], is devoted to showing that the *flip mapping* that takes $T_{\mathbf{a}, \mathbf{b}} \rightarrow T_{\mathbf{b}, \mathbf{a}}$ is not necessarily continuous. Since the morphisms in the category of operator spaces are the completely bounded mappings he was familiar at this stage with associated concepts such as complete positivity and it is not surprising that he then attacked more fundamental and difficult problems about elementary operators. His investigations can be divided into three main topics: complete positivity, computation of the norm, and applications to the ideal theory of $\mathcal{C}^*$ -algebras. There are a number of surveys on the progress achieved by elementary operators at different times, e.g. [A2], [M2] and [M3], and we also refer to [32] for an up-to-date guide to the literature.

Richard's two main contributions to positivity of elementary operators on $\mathcal{C}^*$ -algebras are [23] and [25]. In [23], he examined in turn the situation for $M_n(\mathbb{C})$, $\mathcal{B}(\mathcal{H})$ and an arbitrary $\mathcal{C}^*$ -algebra, and proved the following (best possible) result.

Theorem 19. *Let A be a $\mathcal{C}^*$ -algebra, and let T be an elementary operator on A of minimal length at most $(k+1)^2 - 1$. If T is k -positive, then T is completely positive.*

It is interesting to follow Richard's successive generalisations as he proved and extended this result. He proved that k -positivity was equivalent to showing that a certain quadratic form was positive on elements of rank k . For each generalisation it was necessary to find the appropriate definition of finite rank and rank k , and to show that the quadratic form was well defined, i.e., that it did not depend on the representation of the elementary operator. Afterwards, he showed that the whole process was *natural*. All these required very insightful matrix calculations. He began, in [23], by using the fact that every elementary operator on $M_n(\mathbb{C})$ that maps self-adjoint to self-adjoint can be written as $Tx = \sum_{i=1}^l \lambda_i a_i x a_i^*$, where

$\lambda_i \in \mathbb{R}$. This led him to the real-valued quadratic form Q_T on $M_n(\mathbb{C})$:

$$Q_T(z) = \sum_{i=1}^l \lambda_i |\text{trace}(a_i z^*)|^2.$$

For the elementary operator, $T(z) = \sum_{i=1}^l a_i z b_i$ on $\mathcal{B}(\mathcal{H})$, this was modified in [25] to define a real-valued quadratic form on $\mathcal{F}(\mathcal{H})$, the finite rank operators in $\mathcal{B}(\mathcal{H})$,

$$Q_T(z) = \sum_{i=1}^l \text{trace}(a_i z^*) \text{trace}(b_i z).$$

On a $\mathcal{C}^*$ -algebra A , he defined an element $\phi \in A^*$ to have *finite rank* and *singleton support* if there exists an irreducible $*$ -representation $\pi : A \rightarrow \mathcal{B}(\mathcal{H}_\pi)$ and a finite rank operator $y_\phi : \mathcal{H}_\pi \rightarrow \mathcal{H}_\pi$ such that $\phi(x) = \text{trace}(\pi(x)y_\phi)$. Let $\mathcal{F}(A)$ denote the set of all such functionals. The quadratic form corresponding to the elementary operator $T_{\mathbf{a}, \mathbf{b}}$ is now defined on $\mathcal{F}(A)$ by the formula

$$Q_{T_{\mathbf{a}, \mathbf{b}}}(\phi) = \sum_{i=1}^l \overline{\phi(a_i^*)} \phi(b_i).$$

We now consider the problem of computing the norm of an elementary operator. Two of Richard's important contributions to the computation problem were his introduction and application of the *matrix numerical range* and the *tracial geometric mean*. The matrix numerical range was motivated by results of Stampfli in [S1], who proved, using the classical numerical range, that $\|\delta_a\| = 2 \inf_{\lambda \in \mathbb{C}} \|a - \lambda \mathbf{1}\|$ when $a \in \mathcal{B}(\mathcal{H})$ and $\delta_a = L_a - R_a$. If $\mathbf{c} := (c_1, \dots, c_l)$ and each $c_i \in \mathcal{B}(\mathcal{H})$, then the *matrix numerical range* $W_m(\mathbf{c})$ is defined as follows:

$$W_m(\mathbf{c}) = \{(\langle c_j^* c_i \xi, \xi \rangle)_{i,j=1}^l : \xi \in \mathcal{H}, \|\xi\| = 1\} \subset M_l(\mathbb{C}),$$

and the *extremal matrix numerical range* is defined as

$$W_{m,e}(\mathbf{c}) = \{\alpha \in \overline{W_m(\mathbf{c})} : \text{trace}(\alpha) = \|\sum_{i=1}^l c_i^* c_i\|\}.$$

The following results and many concrete examples and extensions to particular $\mathcal{C}^*$ -algebras are presented in [26]. The convex hull is denoted by co .

Theorem 20. *Let $T_{\mathbf{a}, \mathbf{b}}$ denote an elementary operator on $\mathcal{B}(\mathcal{H})$. Then*

$$\|T_{\mathbf{a}, \mathbf{b}}\| \leq \frac{1}{2} \left(\left\| \sum_{j=1}^l a_j a_j^* \right\| + \left\| \sum_{j=1}^l b_j^* b_j \right\| \right)$$

and we have equality if and only if

$$W_{M,e}(\mathbf{a}^*) \cap W_{M,e}(\mathbf{b}^*) \neq \emptyset.$$

Moreover,

$$\|T_{\mathbf{a},\mathbf{b}}\|_{cb} \leq \frac{1}{2} \left(\left\| \sum_{j=1}^l a_j a_j^* \right\| + \left\| \sum_{j=1}^l b_j^* b_j \right\| \right)$$

and we have equality if and only if

$$\text{co}(W_{M,e}(\mathbf{a}^*)) \cap \text{co}(W_{M,e}(\mathbf{b}^*)) \neq \emptyset.$$

Also, $\|T_{\mathbf{a},\mathbf{b}}\|_{cb} = \|T_{\mathbf{a},\mathbf{b}}^{(k)}\|$ with $k = \min(l, \dim(\mathcal{H}))$.

In the following theorem a mapping $T : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is *self-adjoint* if $(T(x^*))^* = T(x)$ for all $x \in \mathcal{B}(\mathcal{H})$.

Theorem 21. *If the elementary operator $T : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ is self-adjoint and has minimal length l , then T can be written as*

$$Tx = \sum_{j=1}^m b_j^* x b_j - \sum_{j=m+1}^l b_j^* x b_j$$

with $\|T\|_{cb} = \|\sum_{j=1}^l b_j^* b_j\|$. If $k = [\sqrt{1 + 2m(l-m)}]$, where $[\]$ denotes integer part, then $\|T^{(k)}\| = \|T\|_{cb}$.

In [27], Richard settled a conjecture of Mathieu [M3] by showing that $\|T\| \geq \|a\| \cdot \|b\|$ when $a, b \in \mathcal{B}(\mathcal{H})$ and $T(x) = axb + bxa$ for $x \in \mathcal{B}(\mathcal{H})$.

If A is an algebra, $\mathbf{a} = (a_1, \dots, a_l)$ and $\mathbf{b} = (b_1, \dots, b_l)$ are l -tuples of elements of A , let

$$\mathbf{a} \odot \mathbf{b} = \sum_{i=1}^l a_i \otimes b_i.$$

The *Haagerup tensor norm* $\|\cdot\|_h$ (see [H1]) on $A \otimes A$ is defined by letting

$$\|\mathbf{a} \odot \mathbf{b}\|_h^2 = \inf \left\{ \sum_{j=1}^k \|c_j c_j^*\| \cdot \sum_{j=1}^k \|d_j^* d_j\| : \mathbf{c} \odot \mathbf{d} = \mathbf{a} \odot \mathbf{b} \right\}.$$

A $\mathcal{C}^*$ -algebra is *prime* if $J_1 \cap J_2 = \{0\}$ for ideals J_1 and J_2 implies $J_1 = \{0\}$ or $J_2 = \{0\}$. Let $(\mathcal{CB}(A), \|\cdot\|_{cb})$ denote the Banach space of all completely bounded mappings from A to itself with the $\|\cdot\|_{cb}$ norm. It is known that the map θ , given by $\theta(\mathbf{a} \odot \mathbf{b}) = T_{\mathbf{a},\mathbf{b}}$, is an isometry from $(A \otimes A, \|\cdot\|_h)$ onto its image in $(\mathcal{CB}(A), \|\cdot\|_{cb})$ if and only if A is a prime $\mathcal{C}^*$ -algebra (see [A2, 5.4.11], [H1] and [M5, p. 158]).

For an arbitrary C^* -algebra, let J denote the closed ideal generated by the elements $az \otimes b - a \otimes zb$, z in the centre of A , and let $A \otimes_{Z,h} A = (A \otimes_h A)/J$ denote the *central Haagerup tensor product*. In [28] the authors consider the mapping

$$\theta_{Z,h} : A \otimes_{Z,h} A \longrightarrow \mathcal{CB}(A), \quad \theta_{Z,h}(\mathbf{a} \odot \mathbf{b} + J) = T_{\mathbf{a},\mathbf{b}}.$$

To present their results, which show a relationship between isometric and algebraic properties, we need some definitions. If M is a maximal ideal in the centre of a unital C^* -algebra A , we call AM the *Glimm ideal* generated by M . A closed two-sided ideal I is called *n-primal* if whenever $J_1, \dots, J_n$ are ideals in A and $J_1 \cdots J_n = \{0\}$, then at least one J_i is contained in I , and I is *primal* if it is *n-primal* for all n . Theorem 22 is proved in [28] and extended in a highly technical fashion to the non-unital case in [31].

Theorem 22. *Let A denote a unital C^* -algebra. Then:*

- (1) *The mapping $\theta_{Z,h}$ is an isometry if and only if every Glimm ideal in A is primal.*
- (2) *If $l \geq 1$, then $\|\theta_{Z,h}(u)\|_{cb} = \|u\|_{Z,h}$ for every $u = \sum_{i=1}^l a_i \otimes b_i$ if and only if every Glimm ideal is $(l^2 + 1)$ -primal.*

In [29] Richard introduced the concept of *tracial geometric mean* of two positive semi-definite X and Y in $M_l(\mathbb{C})$ by letting

$$tgm(X, Y) = \text{trace}[(X^{1/2}YX^{1/2})^{1/2}]$$

and proved the following result.

Theorem 23. *If $\mathbf{a}$ and $\mathbf{b}$ are l -tuples in $\mathcal{B}(\mathcal{H})$, then*

- (1) $\|T_{\mathbf{a},\mathbf{b}}\| = \sup\{tgm(X, Y) : X \in W_m(\mathbf{a}^*), Y \in W_m(\mathbf{b})\},$
- (2) $\|T_{\mathbf{a},\mathbf{b}}\|_{cb} = \sup\{tgm(X, Y) : X \in co(W_m(\mathbf{a}^*)), Y \in co(W_m(\mathbf{b}))\}.$

The tracial geometric mean and irreducible representations are employed in [30] to provide new proofs of results and characterisations involving elementary operators. For a representation $(\mathcal{H}_\pi, \pi)$ and an elementary operator $T = T_{\mathbf{a},\mathbf{b}}$ on a C^* -algebra A , let $T^\pi = T_{\pi(\mathbf{a}),\pi(\mathbf{b})}$ denote the associated elementary operator on $\mathcal{B}(\mathcal{H}_\pi)$. Conditions on A are given so that for certain $k > n$, n -positivity of T implies k -positivity of T^π .

Ideals that are the kernels of irreducible representations are called *primitive ideals*, and $\text{Prim}(A)$ denotes the set of all primitive ideals in the C^* -algebra A (see [M5, p. 156]). If $E \subset \text{Prim}(A)$ is non-empty, let

$$\text{hull}(\ker(E)) = \{S \in \text{Prim}(A) : (\bigcap_{R \in E} R) \subset S\}.$$

The sets $\text{hull}(\ker(E))$ are the closed sets for the *hull-kernel topology* on $\text{Prim}(A)$

(see [M5, pp. 157–159]). A C^* -algebra A isomorphic to the set of all continuous sections vanishing at infinity of a locally trivial bundle $\mathfrak{E}$ over $\text{Prim}(A)$ with fibre $M_n(\mathbb{C})$ is called n -homogeneous. Let $El_1(A)$ be the set of all elementary operators of minimal length 1 on the C^* -algebra A . The authors first prove in [33] that $El_1(A)$ is a closed subset of $\mathcal{B}(A)$ when A is a prime C^* -algebra, and then prove the following result.

Theorem 24. *If A is an n -homogeneous C^* -algebra, then $El_1(A)$ is not closed in $\mathcal{B}(A)$ if and only if there exists a σ -compact open subset U of $\text{Prim}(A)$ and a locally trivial line sub-bundle of $\mathfrak{E}$ (as above) that is not globally trivial on U but that is trivial over every compact subset of U .*

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Operator spaces

The interplay between different special subspaces of $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, was the topic of Richard’s last major set of mathematical investigations. His previous research involving JB^* -triples, C^* -algebras, the matrix spaces $M_n(\mathbb{C})$ and his experience with completely bounded operators were important for this study, which also involved two new collections: ternary rings of operators and operator spaces. Many preliminary results in [34] evolved into final results in the later papers, and it is interesting to observe how the experience gained by examining first the Cartan factors helped guide the authors towards their definitive results. For background information we refer to [E1, H3, H5, K1, P1].

A *ternary ring of operators* (TRO) is a closed subspace E of $\mathcal{B}(\mathcal{H})$ such that $[x, y, z] := xy^*z \in E$ whenever $x, y, z \in E$. Since a JC^* -triple is a closed subspace F of $\mathcal{B}(\mathcal{H})$ for which $\{x, y, z\} = (xy^*z + zy^*x)/2 \in F$ whenever $x, y, z \in F$, we see that every TRO is a JC^* -triple.

In [34, 35, 36, 37], an important role is played by certain universal objects. If E is a JC^* -triple, then the *universal* TRO of E is a pair $(T^*(E), \alpha_E)$, where $T^*(E)$ is a TRO and $\alpha_E : E \rightarrow T^*(E)$ is an injective triple homomorphism such that $\alpha_E(E)$ generates $T^*(E)$ as a TRO, and for each triple homomorphism $\pi : E \rightarrow T$, T a TRO, there is a unique TRO morphism $\tilde{\pi} : T^*(E) \rightarrow T$ such that $\tilde{\pi} \circ \alpha_E = \pi$. Universal TROs exist and are unique up to TRO isomorphisms.

In general, given elements $(a_1, \dots, a_{2n+1})$ in a JC^* -subtriple E of $\mathcal{B}(\mathcal{H})$, the reversible element

$$a_1 a_2^* a_3 a_4^* \cdots a_{2n}^* a_{2n+1} + a_{2n+1} a_{2n}^* a_{2n-1} \cdots a_2^* a_1 \quad (*)$$

will exist in $\mathcal{B}(\mathcal{H})$ but may not belong to E . If E contains all reversible elements arising in this way, then it is said to be *reversible* in $\mathcal{B}(\mathcal{H})$. A JC^* -triple E is *universally reversible* if $\pi(E)$ is reversible in $\mathcal{B}(\mathcal{H})$ for every triple homomorphism $\pi : E \rightarrow \mathcal{B}(\mathcal{H})$. Equivalently, E is universally reversible if $\alpha_E(E)$ is reversible in $T^*(E)$. By theorem 4.17 in [37], $(*)$ need only be verified by 5 elements in $\alpha_E(E)$ (that is, one can take $n = 2$). Universally reversible JC^* -triples form a class of Banach spaces preserved by linear isometries.

The authors prove a great variety of results in their investigations. Some are quite technical, relying on structural properties of $\mathcal{C}^*$ -algebras and JB^* -triples, and we confine ourselves to results that may be easily presented. The following result is proved in [34].

Theorem 25. *Spin factors of dimension greater than 4, and Hilbert spaces of dimension greater than 2, are not universally reversible. All other Cartan factors are universally reversible.*

In [37], this result is extended and the following proved.

Theorem 26. *A JC^* -triple E is universally reversible if and only if it satisfies both of the following conditions:*

- a. E has no representation onto a Hilbert space of dimension at least 3,
- b. E has no representation onto a spin factor of dimension at least 5.

Operator spaces are defined as pairs (X, i) where X is a Banach space and i is an embedding of X in $\mathcal{B}(\mathcal{H})$. Surprisingly, the same mappings, complete isometries, characterise isomorphisms in the two categories, as the following is true.

Theorem 27. *If the operator spaces (X, i) and (Y, j) are TROs, then the following are equivalent:*

- (1) (X, i) and (Y, j) are isomorphic operator spaces,
- (2) (X, i) and (Y, j) are isomorphic TROs,
- (3) *there exists an invertible linear mapping T from X onto Y such that $\|T\|_{cb} = \|T^{-1}\|_{cb} = 1$.*

This is in contrast to the situation for JC^* -triples (see Proposition 1), where only an isometry is required. The JC -operator structures on a JC^* -triple E are the operator structures induced by injective triple homomorphisms $\pi : E \rightarrow T$, T a TRO, and a JC -operator space is a JC^* -triple together with a JC -operator structure. Indeed, as remarked in [34]: *The fundamental difference between JC^* -triples and TROs is that the underlying structure of the former is determined by their linear isometric class, while that of the latter is fixed up to complete isometry.*

A Hilbert space $\mathcal{H}$ with inner-product $\langle \cdot, \cdot \rangle$ becomes a JC^* -triple when endowed with the triple product $\{x, y, z\} = (\langle x, y \rangle z + \langle z, y \rangle x)/2$. For x a fixed unit vector in the infinite-dimensional separable Hilbert space $\mathcal{H}$, let $\mathbb{C}x = \{\lambda x : \lambda \in \mathbb{C}\}$ and let $x^\perp = \{y \in \mathcal{H} : \langle x, y \rangle = 0\}$. Consider the following two subspaces of $\mathcal{B}(\mathcal{H})$: $\mathcal{H}_c := \{T \in \mathcal{B}(\mathcal{H}) : T(\mathcal{H}) \subset \mathbb{C}x\}$ and $\mathcal{H}_r := \{T \in \mathcal{B}(\mathcal{H}) : T(x^\perp) = \{0\}\}$. These spaces, and the notation, become more transparent when we choose an orthonormal basis $(e_n)_{n=1}^\infty$ for $\mathcal{H}$ with $x = e_1$, and identify $\mathcal{B}(\mathcal{H})$ with the set of all matrices $(\langle T(e_m), e_n \rangle)_{m,n=1}^\infty$. In this setting, $\mathcal{H}_c$ (respectively $\mathcal{H}_r$) is identified with the matrices whose non-zero entries lie in the first column (respectively the first row), and is isometrically isomorphic to the Hilbert space $\mathcal{L}(\mathcal{H}, \mathbb{C})$ (respectively $\mathcal{L}(\mathbb{C}, \mathcal{H})$). The embedding of $\mathcal{H}_c$ and $\mathcal{H}_r$ in $\mathcal{B}(\mathcal{H})$ endows both of these spaces with an operator space structure, but these isometric Hilbert spaces are not isomorphic as operator spaces. Hilbertian operator spaces that are TROs are completely isometric to either $\mathcal{H}_c$ or $\mathcal{H}_r$. In [35] the authors provide a full description of the JC -operator spaces on a Hilbert space of arbitrary dimension and show that every Hilbertian operator space X is *homogeneous*, that is, if $T \in \mathcal{B}(X)$ then T is completely bounded and $\|T\| = \|T\|_{cb}$.

Associative JC^* -triples, previously considered in Theorem 13, are precisely the triples that coincide with their own universal TROs. This is part of the following result from [37].

Theorem 28. *The following are equivalent for a JC^* -subtriple E of $\mathcal{B}(\mathcal{H})$.*

- (1) E is an associative JC^* -triple.
- (2) E is completely isometric to a sub-TRO of an Abelian C^* -algebra.
- (3) E has a separating family of representations onto $\mathbb{C}$.
- (4) $T^*(E) = E$.

A closed subspace I of a JC^* -triple is an *ideal* if $\{x, y, z\} \in I$ whenever one of $x, y, z \in I$. Every C^* -algebra is a TRO and every TRO is a JC^* -triple, and all

three structures generate (or define) the same ideals (see lemma 2.1 in [34]). Let E be a JC^* -triple. Two JC -operator space structures E_1 and E_2 on E are said to be equal if and only if the identity map $E_1 \rightarrow E_2$ is a complete isometry. An *operator space ideal* in $T^*(E)$ is a (norm-closed) ideal $\mathcal{I}$ of $T^*(E)$ such that $\alpha_E(E) \cap \mathcal{I} = 0$. For each operator space ideal $\mathcal{I}$ of $T^*(E)$ we have the JC -operator space structure, $E_{\mathcal{I}}$, on E determined by the isometric embedding $E \rightarrow T^*(E)/\mathcal{I}$ where $x \rightarrow \alpha_E(x) + \mathcal{I}$. Denote the image by $\widetilde{E_{\mathcal{I}}}$. Note that $TRO(\widetilde{E_{\mathcal{I}}}) = T^*(E)/\mathcal{I}$. Given operator space ideals $\mathcal{I}$ and $\mathcal{J}$ in $T^*(E)$, $\widetilde{E_{\mathcal{I}}} = \widetilde{E_{\mathcal{J}}}$ if and only if the mapping $\alpha_E(x) + \mathcal{I} \rightarrow \alpha_E(x) + \mathcal{J}$ is a complete isometry. By the functorial properties of the universal TRO, if $\pi : E \rightarrow F$ is a linear isometry onto a JC^* -triple F , then $T^*(\pi) : T^*(E) \rightarrow T^*(F)$ preserves operator space ideals.

The following results are proved in [34] and [36].

Theorem 29. (1) *Let $\mathcal{I}$ be an ideal in a JC^* -triple E . Then E is universally reversible if and only if $\mathcal{I}$ and $E/\mathcal{I}$ are both universally reversible.*
 (2) *Let E be a JC^* -triple. Then E contains a largest universally reversible ideal $\mathcal{I}$, and $E/\mathcal{I}$ has no non-zero universally reversible ideals.*
 (3) *Let E be a JC^* -triple and let $\pi : E \rightarrow F$ denote a linear isometry onto a JC^* -subtriple F of $\mathcal{B}(\mathcal{H})$ (regarded as an operator subspace of $\mathcal{B}(\mathcal{H})$). Then there is an operator space ideal $\mathcal{I}$ of $T^*(E)$ such that $\pi : E_{\mathcal{I}} \rightarrow F$ is a complete isometry. Hence, every JC -operator space structure on E equals $E_{\mathcal{I}}$ for some operator space ideal $\mathcal{I}$.*
 (4) *If E is a universally reversible JC^* -triple or a Cartan factor, then $\mathcal{I} \longleftrightarrow E_{\mathcal{I}}$ is a bijective correspondence between the operator space ideals of $T^*(E)$ and the JC operator space structures of E .*

We have seen that Hilbert spaces may have more than one non-isomorphic operator space structure. On the other hand, $\mathcal{C}^*$ -algebras and TROs have unique operator space structures. The following result, from [36], considers the JC^* -triple situation.

Theorem 30. *The following are equivalent conditions for a universally reversible JC^* -triple E .*

- (1) *E has a unique JC -operator space structure.*
- (2) *$T^*(E)$ has no non-zero operator space ideal.*
- (3) *E has no ideal linearly isometric to a non-Abelian TRO.*
- (4) *If $\pi : E \rightarrow \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, is an injective triple homomorphism, then $(TRO(\pi(E)), \pi)$ is the universal TRO of E .*

If X and Y are Banach spaces and $\sum_{i=1}^n x_i \otimes y_i \in X \otimes Y$, let

$$\left\| \sum_{i=1}^n x_i \otimes y_i \right\|_{\varepsilon} = \sup \left\{ \left| \sum_{i=1}^n \phi(x_i) \psi(y_i) \right| : \phi \in B_{X'}, \psi \in B_{Y'} \right\},$$

and let $X \hat{\otimes}_\varepsilon Y$ denote the completion of $(X \otimes Y, \|\cdot\|_\varepsilon)$. For example, if K is a compact Hausdorff space, then the space of Y -valued continuous functions on K has the following representation: $\mathcal{C}(K) \hat{\otimes}_\varepsilon Y \simeq \mathcal{C}(K, Y)$ (see [R1, p. 49]). In [36] the authors prove the following result.

Theorem 31. *Let $\mathcal{A}$ be an Abelian C^* -algebra and E a JC^* -triple. Then,*

- (1) $T^*(\mathcal{A} \hat{\otimes}_\varepsilon E) = \mathcal{A} \hat{\otimes}_\varepsilon T^*(E)$,
- (2) $\mathcal{A} \hat{\otimes}_\varepsilon E$ is universally reversible if and only if E is universally reversible.

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Miscellaneous topics

1. Dunford-Pettis property. A Banach space X has the *Dunford-Pettis property* (DPP) if whenever $(x_n)_{n=1}^\infty$ is a weakly null sequence in X and $(x_n^*)_{n=1}^\infty$ is a weakly null sequence in X' , then $(x_n^*(x_n))_{n=1}^\infty$ is a null sequence of complex numbers. We refer to [D3] for a survey article on the DPP. While visiting the University of North Carolina during the academic year 1985–86, Richard wrote a joint article with Joe Cima [38] on the (DPP). Subsequently, Richard gave a seminar in Paris and included some additional results in his report [39]. We present their result for smooth domains as Theorem 32 below, and note that the general, that is not-necessarily smooth, case may be proved using the Henkin kernel (see [H8]) and following the instructions outlined in [39]. For smooth domains we need the following definition from [K3]. A bounded domain $\mathcal{D}$ in $\mathbb{C}^n$ is *strictly pseudo-convex* if there exists a $\mathcal{C}^2$ function $\rho : U \rightarrow \mathbb{R}$ defined on a neighbourhood U of the boundary of $\mathcal{D}$, $\partial\mathcal{D}$, such that $\mathcal{D} \cap U = \{z \in U : \rho(z) < 0\}$, $\partial\mathcal{D} = \{z \in U : \rho(z) = 0\}$, $\rho'(z) \neq 0$ all $z \in \partial\mathcal{D}$ and

$$\sum_{i,j=1}^n \frac{\partial^2 \rho(p)}{\partial z_i \partial \bar{z}_j} w_i \bar{w}_j > 0$$

for all $p \in U$ and all non-zero $(w_i)_{i=1}^n \in \mathbb{C}^n$.

Theorem 32. *If D is a strictly pseudo-convex bounded domain in $\mathbb{C}^n$, then*

$$A(\overline{D}) := \{f \in \mathcal{C}(\overline{D}) : f|_D \in \mathcal{H}(\mathcal{D})\}$$

and its dual have the DPP.

2. A fixed point theorem. If $f \in \mathcal{H}(\Delta, \Delta)$ does not have a fixed point, then a

1926 result of Wolff (see [D2, p. 194]) shows that there exists a unique point u on the boundary of Δ such that every disc in Δ , internally tangent to the unit circle at u , is f invariant. The point u may be regarded as a *virtual fixed point* of f , and the motivation in [40] was to extend f in a holomorphic fashion to a natural compactification of Δ so that the virtual fixed point became a *concrete fixed point*. Let $\mathcal{M}$ denote the set of non-zero multiplicative linear functionals on the commutative unital uniform Banach algebra $\mathcal{H}^\infty(\Delta)$. A celebrated result of Carleson, the corona theorem, states that Δ is dense in $\mathcal{M}$ when $\mathcal{M}$ is endowed with the weak*-topology inherited from $(\mathcal{H}^\infty(\Delta))'$ and points in Δ are identified with point evaluations on $\mathcal{H}^\infty(\Delta)$. The composition operator $T_f(g) = g \circ f$ from $\mathcal{H}^\infty(\Delta)$ to $\mathcal{H}^\infty(\Delta)$ is an algebra homomorphism, and hence the transpose T_f^* defined by

$$[T_f^* \phi](g) = \phi(T_f(g)) = \phi(g \circ f),$$

where $\phi \in (\mathcal{H}^\infty(\Delta))'$ and $g \in \mathcal{H}^\infty(\Delta)$, maps $\mathcal{M}$ into $\mathcal{M}$. The identification $z \in \Delta \longleftrightarrow \delta_z \in \mathcal{M}$ implies $T_f^*(\delta_z) = \delta_{f(z)} \longleftrightarrow f(z)$ and $f^\bullet : m \in \mathcal{M} \longrightarrow T_f^*(m)$ extends f . Since T_f^* is weak*-weak* continuous, $f^\bullet$ is the unique continuous extension of f to a self-map of $(\mathcal{M}, \text{weak}^*)$.

Points $\phi, \varphi \in \mathcal{M}$ lie in the same *Gleason part* if $\|\phi - \varphi\| < 2$ and the *Gelfand transform* of $g \in \mathcal{H}^\infty(\Delta)$ is the mapping $\hat{g} : \mathcal{M} \longrightarrow \mathbb{C}$ given by $\hat{g}(\phi) = \phi(g)$. A Gleason part P contains more than one point if and only if it is an *analytic disc*, that is, there exists a bijective mapping $\varphi : \Delta \longrightarrow P$ such that $k \circ \varphi \in \mathcal{H}(\Delta)$ for all $k \in \mathcal{H}^\infty(\Delta)$ (see [G1, p. 400]). Results on finite-dimensional complex manifolds are first proved in [40], and then specialised to prove the following result.

Theorem 33. *If $f^\bullet$ is the unique weak*-to-weak* continuous extension of $f \in \mathcal{H}^\infty(\Delta)$ to the maximal ideal space $\mathcal{M}$ of $\mathcal{H}^\infty(\Delta)$, then either $f^\bullet$ has a fixed point in $\mathcal{M}$ or there exists an analytic disc $\varphi : \Delta \longrightarrow P$ in $\mathcal{M}$ such that $f^\bullet \in \text{Aut}(P)$ and the Möbius transformation $\varphi^{-1} \circ f^\bullet \circ \varphi$ is hyperbolic (that is, it has two fixed points on $\mathbb{T}$, one attracting and one repelling).*

3. *The Ando–von Neumann inequality.* If T_1 and T_2 are a commuting pair of contractions in $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ a Hilbert space, and p is a polynomial in two variables bounded by 1 on Δ^2 , then the Ando–von Neumann inequality states that $p(T_1, T_2)$ is a contraction. It is known that the obvious extension to three or more variables fails. If $\mathbf{T} = (T_1, \dots, T_n)$ is an n -tuple of operators on $\mathcal{B}(\mathcal{H})$, and $z^m = z_1^{m_1} \dots z_n^{m_n}$ is a monomial in n -variables, let $\mathbf{T}^m$ denote the non-commutative polynomial obtained by averaging over all possible ways of writing it with non-commuting variables. For example, if $\mathbf{T} = (T_1, T_2)$ and $z^{(1,2)} = z_1 z_2^2$, then

$$\mathbf{T}^{(1,2)} = (T_1 T_2^2 + T_2 T_1 T_2 + T_2^2 T_1)/3.$$

If $p(z) = \sum_{m \in \mathbb{N}^n} a_m z^m$ is a polynomial, let $p_{\text{sym}}(\mathbf{T}) = \sum_{m \in \mathbb{N}^n} a_m \mathbf{T}^m$. If $\mathbf{T}$ is a commuting n -tuple, that is, if $T_i T_j = T_j T_i$ for $1 \leq i, j \leq n$, then $\mathbf{T}^m = T_1^{m_1} \dots T_n^{m_n}$ and we write $p(\mathbf{T})$. In [43] the authors prove the following two generalisations of

the Ando–von Neumann inequality – the first of which answers for a commuting n -tuple what is called a *stubborn open problem* in [D4, p. 395].

Theorem 34. Let $\mathbf{T} = (T_1, \dots, T_n)$ denote an n -tuple of contractions on $\mathcal{B}(\mathcal{H})$.

(1) For each n there exist constants R_n and M_n such that

$$\|p_{\text{sym}}(\mathbf{T})\| \leq \inf \{ \|p\|_{R_n \Delta^n}, M_n \|p\|_{\Delta^n} \}.$$

(2) If $\mathbf{T}$ is a commuting n -tuple, then $r_\sigma(p(\mathbf{T})) \leq \|p\|_{\Delta^n}$.

4. *Ordinary differential equations.* Using fixed point theory for cones in Banach spaces, the authors in [41] and [42] establish existence results for positive solutions to some singular second-order boundary value problems.

5. *Spaceable sets.* A subset S in a locally convex space X is *spaceable* if $S \cup \{0\}$ contains a closed infinite-dimensional subspace of X , and a subspace Y of X is an *operator range subspace* if there exists a continuous linear operator $T : Z \rightarrow X$ such that $T(Z) = Y$. Operator range subspaces need not be closed, but do exhibit properties similar to those of closed subspaces. The authors in [44] prove the following result.

Theorem 35. If Y is a non-closed operator range subspace of a Banach space X , then $X \setminus Y$ is spaceable.

Using Theorem 34 and the mapping

$$(a_n)_{n=1}^\infty \in \ell_1 \longrightarrow \{z \in \Delta \longrightarrow \sum_{n=0}^\infty a_n z^n\},$$

they show that the complement of the space of absolutely convergent power series in the disc algebra is spaceable.

6. *Module mapping.* Let $\mathcal{K}(\mathcal{H})$ denote the subspace of compact operators in $\mathcal{B}(\mathcal{H})$, $\mathcal{H}$ an infinite-dimensional Hilbert space, and let D denote a *maximal Abelian sub-algebra* (MASA) of $\mathcal{B}(\mathcal{H})$. When $\mathcal{H}$ is separable, and operators in $\mathcal{B}(\mathcal{H})$ are represented by matrices in the usual way, then the set of all diagonal operators is a MASA. If $T \in \mathcal{B}(\mathcal{K}(\mathcal{H}))$ is a *continuous D -module mapping*, that is, T is continuous and $T(xay) = xT(a)y$ for $x, y \in D$ and $a \in \mathcal{K}(\mathcal{H})$, then it is known that $\|T\| = \|T\|_{cb}$. In [45] the authors show that this is no longer true when T is only a *right module mapping*, that is, $T(ay) = T(a)y$ for $y \in D$ and $a \in \mathcal{K}(\mathcal{H})$. They prove, in fact, that T may not even be completely bounded in this situation. This answers an open problem in [J1].

7. *Non-commutative complex analysis.* If d is a positive integer, let $\mathbb{M}_n^d$ denote the d -tuples of elements in $M_n(\mathbb{C})$, and let $\mathbb{M}^{[d]}$ denote the disjoint union $\bigcup_{n=1}^\infty \mathbb{M}_n^d$. If $x = (x^i)_{i=1}^d \in \mathbb{M}_n^d$ and $y = (y^i)_{i=1}^d \in \mathbb{M}_m^d$, let $x \oplus y = (x^i \oplus y^i)_{i=1}^d \in \mathbb{M}_{n+m}^d$,

and if $s, t \in M_n(\mathbb{C})$, let $sxt = (sx^it)_{i=1}^d$. An *nc-domain* is a set $\Omega \subset \mathbb{M}^{[d]}$ such that $\Omega_n := \Omega \cap \mathbb{M}_n^d$ is a connected open subset of $\mathbb{M}_n^d$ for each n , which is closed with respect to direct sums and such that for all $x \in \Omega_n$ and all unitary $u \in M_n(\mathbb{C})$ we have $u^{-1}xu \in \Omega$. The domain Ω is bounded if Ω_n is bounded for all n . An *nc-map* on an *nc-domain* $\Omega \subset \mathbb{M}^{[d]}$ is a d -tuple, $(f_i)_{i=1}^d$, where each f_i satisfies:

- (1) $f_i(\Omega_n) \subset M_n(\mathbb{C})$,
- (2) if $x, y \in \Omega$ then $f_i(x \oplus y) = f_i(x) \oplus f_i(y)$,
- (3) if $x \in \Omega_n$, $s \in M_n(\mathbb{C})$ is invertible and $s^{-1}xs \in \Omega$ then

$$f_i(s^{-1}xs) = s^{-1}f_i(x)s.$$

A bijective *nc-map* is called an *nc-automorphism*. The following rigidity theorem that generalises Cartan's uniqueness result (see for instance [H5]) is proved in [46].

Theorem 36. *Let Φ denote an nc-automorphism of the bounded nc-domain Ω . If for some m , $0 \in \Omega \cap \mathbb{M}_m^d$, $\Phi|_{\Omega_m}(0) = 0$ and $(\Phi|_{\Omega_m})'(0) = I_{\mathbb{M}_m^d}$, then $\Phi(z) = z$ for all $z \in \Omega$.*

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