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PANEITZ OPERATORS ON HYPERBOLIC SPACES AND HIGH ORDER HARDY-SOBOLEV-MAZ'YA INEQUALITIES ON HALF SPACES

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Abstract. Though there has been extensive study on Hardy-Sobolev-Maz'ya inequalities on upper half spaces for first order derivatives, whether an analogous inequality for higher order derivatives holds has still remained open. By using, among other things, the Fourier analysis techniques on the hyperbolic space which is a noncompact complete Riemannian manifold, we establish the Hardy-Sobolev-Maz'ya inequalities for higher order derivatives on half spaces. Moreover, we derive sharp Poincaré-Sobolev inequalities (namely, Sobolev inequalities with a substraction of a Hardy term) for the Paneitz operators on hyperbolic spaces which are of their independent interests and useful in establishing the sharp Hardy-Sobolev-Maz'ya inequalities. Our sharp Poincaré-Sobolev inequalities for the Paneitz operators on hyperbolic spaces improve substantially those Sobolev inequalities in the literature. The proof of such Poincaré-Sobolev inequalities relies on hard analysis of Green's functions estimates, and Fourier analysis on hyperbolic spaces together with the Hardy-Littlewood-Sobolev inequality on the hyperbolic spaces. Finally, we show the sharp constant in the Hardy-Sobolev-Maz'ya inequality for the bi-Laplacian in the upper half space of dimension five coincides with the best Sobolev constant. This is an analogous result to that of the sharp constant in the first order Hardy-Sobolev-Maz'ya inequality in the three dimensional upper half spaces.

1. Introduction. Let $n \geq 3$, $2 < p \leq \frac{2n}{n-2}$ and $\gamma = \frac{(n-2)p}{2} - n$. The Hardy-Sobolev-Maz'ya inequality on the half space $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) : x_1 > 0\}$ reads as follows (see [51, Section 2.1.6]):

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_1^2} dx \geq C\left(\int_{\mathbb{R}^n_+} x_1^\gamma |u|^p dx\right)^{\frac{2}{p}}, \quad u \in C_0^\infty(\mathbb{R}^n_+), \tag{1.1}$$

where C is a positive constant which is independent of u. In particular, for $\gamma = 0$ and $p = \frac{2n}{n-2}$ we have

$$\int_{\mathbb{R}^n_+} |\nabla u|^2 dx - \frac{1}{4}\int_{\mathbb{R}^n_+} \frac{u^2}{x_1^2} dx \geq C_n\left(\int_{\mathbb{R}^n_+} |u|^{\frac{2n}{n-2}} dx\right)^{\frac{n-2}{2}}, \quad u \in C_0^\infty(\mathbb{R}^n_+). \tag{1.2}$$

It has been shown by R. D. Benguria, R. L. Frank, and M. Loss [10] that the sharp constant C_3 in (1.2) for $n = 3$ coincides with the corresponding best Sobolev constant. In the paper [48], G. Mancini and K. Sandeep showed inequality (1.1)

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is equivalent to the following Poincaré-Sobolev inequalities on hyperbolic space $\mathbb{H}^n\,(n \geq 3)$:

$$(1.3)\quad \int_{\mathbb{H}^n} |\nabla_{\mathbb{H}} u|^2 dV - \frac{(n-1)^2}{4} \int_{\mathbb{H}^n} u^2 dV \geq C \left(\int_{\mathbb{H}^n} |u|^p dV \right)^{\frac{2}{p}}, \quad u \in C_0^\infty(\mathbb{H}^n),$$

where $2 < p \leq \frac{2n}{n-2}$, $\nabla_{\mathbb{H}}$ is the hyperbolic gradient and dV is the hyperbolic volume element. We refer to [11, 12] for Poincaré-Hardy inequalities on $\mathbb{H}^n$ $(n \geq 3)$. In the case $n = 2$, Beckner [7] obtained a sharp Poincaré-Sobolev inequality on the two dimensional hyperbolic space $\mathbb{H}^2$. Furthermore, there holds some first order Trudinger-Moser type inequalities on hyperbolic spaces [42, 43] and Hardy-Trudinger-Moser inequality on $\mathbb{H}^2$ (see [49, 44, 58]) and Hardy-Adams inequality on $\mathbb{H}^4$ and $\mathbb{H}^n$ for all even dimensions $n \geq 4$ [45, 38]. We note that our recent work [45, 38] on Hardy-Adams inequality on four and higher even dimensional hyperbolic spaces can be regarded as a limiting case of our higher order Hardy-Sobolev-Maz'ya inequality in this paper in even dimensions. More recently, J. Li and the authors have succeeded in establishing sharp Adams and Hardy-Adams inequalities of any fractional order in all dimensions [37]. For more information about first order Hardy-Sobolev-Maz'ya in equalities, we refer to [17, 18, 19, 52, 57].

Though there has been an extensive study on first order Hardy-Sobolev-Maz'ya inequalities (1.1) and (1.3) in the past decades, whether an analogous inequality for higher order derivatives holds has still remained open. In this paper we shall show that high order Hardy-Sobolev-Maz'ya inequalities indeed hold.

To state our results, let us introduce some conventions. It is known that hyperbolic space has the Poincaré ball model and the Poincaré half space model and both models are equivalent. We denote by $\mathbb{B}^n$ the Poincaré ball model. It is the unit ball

$$\mathbb{B}^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \,|\, |x| < 1\}$$

equipped with the usual Poincaré metric

$$ds^2 = \frac{4(dx_1^2 + \cdots + dx_n^2)}{(1-|x|^2)^2}.$$

The hyperbolic volume element is $dV = \left(\frac{2}{1-|x|^2}\right)^n dx$ and the distance from the origin to $x \in \mathbb{B}^n$ is $\rho(x) = \log \frac{1+|x|}{1-|x|}$. Then $\mathbb{B}^n$ is a complete noncompact Riemannian manifold of constant negative curvature. The associated Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}} = \frac{1-|x|^2}{4} \left\{ (1-|x|^2) \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} + 2(n-2) \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} \right\}.$$

The spectral gap of $-\Delta_{\mathbb{H}}$ on $L^2(\mathbb{B}^n)$ is $\frac{(n-1)^2}{4}$, i.e.,

$$\int_{\mathbb{B}^n} |\nabla_{\mathbb{H}} u|^2 dV \geq \frac{(n-1)^2}{4} \int_{\mathbb{B}^n} u^2 dV, \quad u \in C_0^\infty(\mathbb{B}^n). \tag{1.4}$$

The GJMS operators on $\mathbb{B}^n$ is defined as follows (see [24, 34])

$$P_k = P_1(P_1+2)\cdot \cdots \cdot (P_1 + k(k-1)), \quad k \in \mathbb{N}, \tag{1.5}$$

where $P_1 = -\Delta_{\mathbb{H}} - \frac{n(n-2)}{4}$ is the conformal Laplacian on $\mathbb{B}^n$. The sharp Sobolev inequalities on $\mathbb{B}^n$ read as follows (see [29] for $k = 1$ and [41] for $2 \leq k < \frac{n}{2}$):

$$\int_{\mathbb{B}^n} (P_k u) u dV \geq S_{n,k} \left(\int_{\mathbb{B}^n} |u|^{\frac{2n}{n-2k}} dV \right)^{\frac{n-2k}{n}}, \quad u \in C_0^\infty(\mathbb{B}^n),\ 1 \leq k < \frac{n}{2}, \tag{1.6}$$

where $S_{n,k}$ is the best k-th order Sobolev constant. We remark that the above sharp Sobolev inequality on hyperbolic spaces (1.6) can also follow from the Sharp Sobolev inequality on spheres due to Beckner [5] (see Section 7 for another proof). We also refer the reader to the recent works on Sobolev type inequalities for Paneitz operators on compact Riemannian manifolds by Hang and Yang [26, 27, 28]. On the other hand, by (1.4) and (1.5), we have the following Poincaré inequality

$$\int_{\mathbb{B}^n} (P_k u) u dV \geq \prod_{i=1}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{B}^n} u^2 dV, \quad u \in C_0^\infty(\mathbb{B}^n). \tag{1.7}$$

As we will show in the proof of Theorem 1.3 (see Section 5), inequality (1.7) is equivalent to the Hardy inequality on the upper half space

$$\int_{\mathbb{R}^n_+} |\nabla^k u|^2 dx \geq \prod_{i=1}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_1^{2k}} dx, \quad u \in C_0^\infty(\mathbb{R}^n_+),$$

and the constant $\prod_{i=1}^{k} \frac{(2i-1)^2}{4}$ is sharp (see [53]).

Next we define another $2k$-th order operator Q_k with $k \geq 2$:

$$\begin{aligned} Q_k &= \left(-\Delta_{\mathbb{H}} - \frac{(n-1)^2}{4}\right)(P_1+2)\cdot \cdots \cdot (P_1 + k(k-1)) \\ &= P_k - \frac{1}{4}(P_1+2)\cdot \cdots \cdot (P_1 + k(k-1)). \end{aligned} \tag{1.8}$$

To this end, we have the following Sobolev type inequalities for Q_k.

THEOREM 1.1. *Let $2 \leq k < \frac{n}{2}$ and $2 < p \leq \frac{2n}{n-2k}$. There exists a positive constant C such that for each $u \in C_0^\infty(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} (Q_k u) u dV \geq C \left(\int_{\mathbb{B}^n} |u|^p dV \right)^{\frac{2}{p}}. \tag{1.9}$$

Notice that, by (1.4),

$$\begin{aligned} &\int_{\mathbb{B}^n} \big((P_1 + 2) \cdot \dots \cdot (P_1 + k(k-1)) u \big) u dV \\ &\geq \prod_{i=2}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{B}^n} u^2 dV, \quad u \in C_0^\infty(\mathbb{B}^n). \end{aligned} \tag{1.10}$$

Combining (1.10), (1.7) and (1.8) yields

$$\int_{\mathbb{B}^n} (P_k u) u dV - \prod_{i=1}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{B}^n} u^2 dV \geq \int_{\mathbb{B}^n} (Q_k u) u dV.$$

We note that the proof of Theorem 1.1 uses the Hardy-Littlewood-Sobolev inequality on hyperbolic spaces (see Theorem 4.1). However, Theorem 4.1 alone is not sufficient to establish the above sharp Sobolev inequality on hyperbolic spaces (i.e., Theorem 1.1). More delicate Green's function estimates for the kernels of powers of fractional Laplacians and Fourier analysis on hyperbolic spaces are required. Sections 3 and 4 devote to such estimates and analysis.

As an application of Theorem 1.1, we have the following Poincaré-Sobolev inequalities for higher order derivatives.

THEOREM 1.2. *Let $2 \leq k < \frac{n}{2}$ and $2 < p \leq \frac{2n}{n-2k}$. There exists a positive constant $C = C(n,p)$ such that for each $u \in C_0^\infty(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} (P_k u) u dV - \prod_{i=1}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{B}^n} u^2 dV \geq C \left(\int_{\mathbb{B}^n} |u|^p dV \right)^{\frac{2}{p}}. \tag{1.11}$$

This improves substantially the Poincaré-Sobolev inequalities (1.6) for higher order derivatives on the hyperbolic spaces $\mathbb{B}^n$ established by Liu [41] (see also Beckner [5]).

If $p = \frac{2n}{n-2k}$, then by (1.6), the sharp constant in (1.11) is less than or equal to the best k-th order Sobolev constant.

As an application of Theorem 1.2, we have the following Hardy-Sobolev-Maz'ya inequalities for higher order derivatives:

THEOREM 1.3. *Let $2 \leq k < \frac{n}{2}$ and $2 < p \leq \frac{2n}{n-2k}$. There exists a positive constant C such that for each $u \in C_0^\infty(\mathbb{R}_+^n)$,*

$$\int_{\mathbb{R}_+^n} |\nabla^k u|^2 dx - \prod_{i=1}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{R}_+^n} \frac{u^2}{x_1^{2k}} dx \geq C \left(\int_{\mathbb{R}_+^n} x_1^\gamma |u|^p dx \right)^{\frac{2}{p}}, \tag{1.12}$$

where $\gamma = \frac{(n-2k)p}{2} - n$.

In terms of the Poincaré ball model $\mathbb{B}^n$, inequality (1.11) can be written as follows:

$$\int_{\mathbb{B}^n} |\nabla^k u|^2 dx - \prod_{i=1}^{k} (2i-1)^2 \int_{\mathbb{B}^n} \frac{u^2}{(1-|x|^2)^{2k}} dx \geq C \left(\int_{\mathbb{B}^n} (1-|x|^2)^\gamma |u|^p dx \right)^{\frac{2}{p}}. \tag{1.13}$$

In some special cases, we can compute the best constant of Poincaré-Sobolev inequalities of fractional order on hyperbolic spaces. The first one is on the hyperbolic spaces of dimension 3 where we have the following:

THEOREM 1.4. *Let $\frac{1}{2} \leq s < \frac{3}{2}$. Then*

$$\int_{\mathbb{B}^3} |(-1-\Delta_{\mathbb{H}})^{\frac{s}{2}} u|^2 dV \geq S_{3,s} \left(\int_{\mathbb{B}^3} |u|^{\frac{6}{3-2s}} dV \right)^{\frac{3-2s}{3}}, \quad u \in C_0^\infty(\mathbb{B}^n),$$

where $S_{3,s} = 2^{2s}\pi^s \frac{\Gamma(\frac{3+2s}{2})}{\Gamma(\frac{3-2s}{2})} \left(\frac{\Gamma(\frac{3}{2})}{\Gamma(3)} \right)^{\frac{2s}{3}}$ is the best Sobolev constant of order s in dimension 3. The inequality is strict for any nonzero u.

Choosing $s = 1$ in Theorem 1.4, we recover the following sharp Poincaré-Sobolev inequality which was first proved by Benguria, Frank and Loss [10]:

$$\int_{\mathbb{B}^3} \left|\nabla_{\mathbb{H}} u\right|^2 dV - \int_{\mathbb{B}^3} |u|^2 dV \geq S_{3,1} \left(\int_{\mathbb{B}^3} |u|^6 dV \right)^{\frac{1}{3}}. \tag{1.14}$$

Though it is not known in general yet whether the constant C on the right-hand sides of the inequalities (1.11), (1.12) and (1.13) in Theorems 1.2 and 1.4 respectively is the same as the best Sobolev constant $S_{n,k}$, we will show in this paper in the case of $n = 5$ and $k = 2$ that the best constant C of Poincaré-Sobolev inequality, as well as the Hardy-Sobolev-Maz'ya inequality in (1.11), (1.12) and (1.13) coincides with the best Sobolev constant $S_{n,k}$. In fact, we have the following:

THEOREM 1.5. *There holds, for each $u \in C_0^\infty(\mathbb{B}^5)$,*

$$\int_{\mathbb{B}^5} (P_2 u) u dV - \frac{9}{16} \int_{\mathbb{B}^5} u^2 dV \geq S_{5,2} \left(\int_{\mathbb{B}^n} |u|^{10} dV \right)^{\frac{1}{5}}. \tag{1.15}$$

The inequality is strict for any nonzero u.

In terms of the Poincaré ball model $\mathbb{B}^5$ and the Poincaré half space model $\mathbb{H}^5$, respectively, inequality (1.15) is equivalent to the follows:

$$\int_{\mathbb{B}^5} |\Delta f|^2 dx - 9\int_{\mathbb{B}^5} \frac{f^2}{(1-|x|^2)^4} dx \geq S_{5,2}\left(\int_{\mathbb{B}^5} |f|^{10} dx\right)^{\frac{1}{5}}, \quad f \in C_0^\infty(\mathbb{B}^5);$$

$$\int_{\mathbb{R}^5_+} |\Delta g|^2 dx - \frac{9}{16}\int_{\mathbb{R}^5_+} \frac{g^2}{x_1^4} dx \geq S_{5,2}\left(\int_{\mathbb{R}^5_+} |g|^{10} dx\right)^{\frac{1}{5}}, \quad g \in C_0^\infty(\mathbb{R}^5_+).$$

Both inequalities are strict for any nonzero f and g.

However, it seems that the best constant for the sharp Poincaré-Sobolev inequality or the Hardy-Sobolev-Maz'ya inequality does not coincide with the best Sobolev constant for $n = 6$ and $k = 2$ (see Remark 8.3). Hong has informed us that in a recent work [32] she proved that the sharp constant of Poincaré-Sobolev inequality and the Hardy-Sobolev-Maz'ya inequality for $n = 7$ and $k = 3$ also coincides with the Sobolev constant $S_{7,3}$, using the same argument as we do in Section 8 of our current paper. We remark that we can continue to show that such constants coincide in higher dimension n by proving it one by one for each given dimension n. However, when n and k are larger, the argument and computation becomes increasingly more difficult. We do not have a uniform proof which covers all the dimensions n and the derivatives of all order k. Nevertheless, through some preliminary work we have been able to make the following conjecture.

CONJECTURE 1.6. *Let $n \geq 9$ be odd. There holds, for each $u \in C_0^\infty(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} \left(P_{(n-1)/2}u\right) u dV - \prod_{i=1}^{(n-1)/2} \frac{(2i-1)^2}{4} \int_{\mathbb{B}^n} u^2 dV \geq S_{n,(n-1)/2}\left(\int_{\mathbb{B}^n} |u|^{2n} dV\right)^{\frac{1}{n}}.$$

In terms of the Poincaré ball model and the Poincaré half space model respectively, the inequality above is equivalent to the following:

$$\int_{\mathbb{B}^n} |\nabla^{\frac{n-1}{2}} f|^2 dx - \prod_{i=1}^{(n-1)/2} (2i-1)^2 \int_{\mathbb{B}^n} \frac{f^2}{(1-|x|^2)^{n-1}} dx$$
$$\geq S_{n,(n-1)/2}\left(\int_{\mathbb{B}^n} |f|^{2n} dx\right)^{\frac{1}{n}}, \quad f \in C_0^\infty(\mathbb{B}^n);$$

$$\int_{\mathbb{R}^n_+} |\nabla^{\frac{n-1}{2}} g|^2 dx - \prod_{i=1}^{(n-1)/2} \frac{(2i-1)^2}{4} \int_{\mathbb{R}^n_+} \frac{g^2}{x_1^{n-1}} dx$$
$$\geq S_{n,(n-1)/2}\left(\int_{\mathbb{R}^n_+} |g|^{2n} dx\right)^{\frac{1}{n}}, \quad g \in C_0^\infty(\mathbb{R}^n_+).$$

We end this introduction with the following remark.

The Hardy-Sobolev-Maz'ya inequality

$$\begin{aligned}\int_{\mathbb{B}^n}(P_k u)u\,dV-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\int_{\mathbb{B}^n}u^2dV\\ \geq C\left(\int_{\mathbb{B}^n}|u|^p dV\right)^{\frac{2}{p}},\quad 2<p\leq\frac{2n}{n-2k},\end{aligned}\tag{1.16}$$

corresponds to the Euler-Lagrange equation:

$$P_k u-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}u=\mu u^{p-1},\tag{1.17}$$

for a constant $\mu>0$. In fact,

$$\mu=\frac{\int_{\mathbb{B}^n}(P_k u)u\,dV-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\int_{\mathbb{B}^n}u^2dV}{\int_{\mathbb{B}^n}|u|^p dV}>0.$$

Note the above equation is equivalent to

$$P_k f-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}f=f^{p-1}.\tag{1.18}$$

By considering the Poincaré half space model of the hyperbolic space and noting the identity (see Lemma 5.1)

$$x_1^{\frac{n}{2}+k}(-\Delta)^k\circ x_1^{k-\frac{n}{2}}=P_k,\tag{1.19}$$

we can see that the above equation can be converted to

$$(-\Delta)^k v-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\frac{v}{x_1^{2k}}=\mu\frac{v^p}{x_1^{\gamma}},\ \ \gamma=\frac{(n-2k)p}{2}-n.\tag{1.20}$$

Using an appropriate transform we can convert μ in the above equation (1.20) to 1, and then convert (1.17) to (1.18).

Therefore, we can consider the following higher order Brezis-Nirenberg problem [13] on the hyperbolic spaces by studying the existence of positive solutions to the following higher order equation

$$P_k u-\lambda u=u^{p-1},\lambda\leq\prod_{i=1}^{k}\frac{(2i-1)^2}{4},2<p\leq\frac{2n}{n-2k}.$$

When $k=1$, the above equation reduces to the one considered in [48]:

$$-\Delta_{\mathbb{H}}u-\left(\frac{n(n-2)}{4}+\lambda\right)u=u^{p-1},\quad \lambda\leq\frac{1}{4},\ 2<p\leq\frac{2n}{n-2}.$$

We note the higher order Brezis-Nirenberg problem in Euclidean spaces has been considered by numerous authors (see e.g., [21, 54] and references therein).

We will return to the study of the above problem in the future.

After the paper was accepted for publication, the authors have confirmed the above conjecture 1.6. Furthermore, the authors have proved in [46] the following Poincaré-Sobolev inequalities on $\mathbb{B}^n$, as well as the Hardy-Sobolev-Maz'ya inequalities on half spaces.

THEOREM 1.7. *Let $2 \leq k < \frac{n}{2}$. Suppose that there exists $\lambda \in \mathbb{R}$ such that for any $u \in C_0^\infty(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} (P_k u) u dV + \lambda \int_{\mathbb{B}^n} u^2 dV \geq S_{n,k} \left(\int_{\mathbb{B}^n} |u|^{\frac{2n}{n-2k}} dV \right)^{\frac{n-2k}{n}} .$$

If $n \geq 4k$, then $\lambda \geq 0$. If $2k+2 \leq n < 4k$, then

$$\lambda \geq -\frac{\Gamma(n/2)\Gamma(k)\sum_{j=0}^{k-1} \frac{\Gamma(j+\frac{n-2k}{2})}{\Gamma(j+1)\Gamma(\frac{n-2k}{2})}}{2^{\frac{n+2k}{2}}\Gamma(\frac{n-2k}{2})\int_0^1 [r^{2k-n} - \sum_{j=0}^{k-1} \frac{\Gamma(j+\frac{n-2k}{2})}{\Gamma(j+1)\Gamma(\frac{n-2k}{2})}(1-r^2)^j]^2 \frac{r^{n-1}dr}{(1-r^2)^{2k}}}. \tag{1.21}$$

In the case of P_1 and $n \geq 4$, the above Theorem 1.7 is in the spirit of the following result due to E. Hebey [29].

THEOREM 1.8. [29] *Let $n \geq 4$. Suppose that there exists $\lambda \in \mathbb{R}$ such that for any $u \in C_0^\infty(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} (P_1 u) u dV + \lambda \int_{\mathbb{B}^n} u^2 dV \geq S_{n,1} \left(\int_{\mathbb{B}^n} |u|^{\frac{2n}{n-2}} dV \right)^{\frac{n-2}{n}} ,$$

then $\lambda \geq 0$.

More recently, by using the Kunze-Stein phenomenon on the closed linear group $\mathrm{SU}(1,n)$ and Fourier analysis techniques on complex hyperbolic spaces, we have established in [47] the Hardy-Sobolev-Maz'ya inequalities on the Siegel domain $\mathcal{U}^n$ and unite ball $\mathbb{B}_{\mathbb{C}}^n$. We have also established the sharp Hardy-Adams inequalities and sharp Adams type inequalities on Sobolev spaces of any positive fractional order on the complex hyperbolic spaces.

The organization of this paper is as follows. In Section 2, we recall some necessary preliminary facts of hyperbolic spaces. Sharp estimates of Green's functions of kernels of certain fractional Laplacians on hyperbolic spaces are given in Section 3. We shall prove Theorem 1.1 and Theorem 1.2 in Section 4. The higher order Hardy-Sobolev-Maz'ya inequalities in all dimension n and all derivatives of order k, namely Theorem 1.3, are proved in Section 5. In Section 6, we prove the sharp constant in Hardy-Sobolev-Maz'ya inequality of fractional order s in case $n = 3$, namely Theorem 1.4. In Section 7, we show that, as in the case of Euclidean space, the Hardy-Littlewood-Sobolev inequlity on hyperbolic space implies the sharp Sobolev inequalities on hyperbolic spaces. This latter result follows from the sharp Sobolev inequality on the sphere obtained by Beckner [5] (see also [41]). In the last section, we prove that the sharp constant for the second order Hardy-Sobolev-Maz'ya inequality in dimension 5 is the same as the best Sobolev constant $S_{5,2}$, namely Theorem 1.5.

Acknowledgments. The authors wish to thank the referees for their very helpful comments and suggestions which improved the exposition of the paper. Indeed, thanks to the insightful comments by the referees, we have discussed in more detail the validity of Theorem 1.5 in general dimension n and included Conjecture 1.6. We have also included a discussion about the relevance of our sharp higher order Hardy-Sobolev-Maz'ya inequalities on hyperbolic spaces to the higher order analogue of the Brezis-Nirenberg problem for the higher order PDEs. The authors also wish to thank W. Beckner for his interest in our work and his encouragement.

2. Notations and preliminaries. We begin by quoting some preliminary facts which will be needed in the sequel and refer to [1, 22, 30, 31, 33, 40] for more information about this subject.

It is well known that hyperbolic space $\mathbb{B}^n$ is a noncompact Riemannian symmetric space of rank one that has a constant negative curvature -1. In fact, $\mathbb{B}^n$ is isomorphic to $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ and in particular, $\mathbb{B}^2$ is also isomorphic to $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$, in which several sharp geometric inequalities have been established by Beckner [6, 7].

There are several models of hyperbolic space, for example, the Poincaré half space model and the Poincaré ball model.

2.1. The Poincaré half space model $\mathbb{H}^n$. It is given by $\mathbb{R}_+ \times \mathbb{R}^{n-1} = \{(x_1, \dots, x_n) : x_1 > 0\}$ equipped with the Riemannian metric $ds^2 = \frac{dx_1^2 + \cdots + dx_n^2}{x_1^2}$. The induced Riemannian measure can be written as $dV = \frac{dx}{x_1^n}$, where dx is the Lebesgue measure on $\mathbb{R}^n$. The hyperbolic gradient is $\nabla_{\mathbb{H}} = x_1 \nabla$ and the Laplace-Beltrami operator on $\mathbb{H}^n$ is given by

$$\Delta_{\mathbb{H}} = x_1^2 \Delta - (n-2) x_1 \frac{\partial}{\partial x_1}, \tag{2.1}$$

where $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplace operator on $\mathbb{R}^n$.

2.2. The Poincaré ball model $\mathbb{B}^n$. It is given by the unit ball

$$\mathbb{B}^n = \left\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid |x| < 1\right\}$$

equipped with the usual Poincaré metric

$$ds^2 = \frac{4(dx_1^2 + \cdots + dx_n^2)}{(1-|x|^2)^2}.$$

The hyperbolic gradient is $\nabla_{\mathbb{H}} = \frac{1-|x|^2}{2}\nabla$ and the Laplace-Beltrami operator is given by

$$\Delta_{\mathbb{H}} = \frac{1-|x|^2}{4}\left\{(1-|x|^2)\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2} + 2(n-2)\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}\right\}.$$

2.3. Möbius transformations. For each $a \in \mathbb{B}^n$, we define the Möbius transformations T_a by (see e.g., [1, 33])

$$T_a(x) = \frac{|x-a|^2 a - (1-|a|^2)(x-a)}{1 - 2x\cdot a + |x|^2|a|^2},$$

where $x \cdot a = x_1 a_1 + x_2 a_2 + \cdots + x_n a_n$ denotes the scalar product in $\mathbb{R}^n$. It is known that the measure on $\mathbb{B}^n$ is invariant with respect to the Möbius transformations. A simple calculation shows

$$\begin{aligned}
T_a\big(T_a(x)\big) &= x;\\
1 - \big|T_a(x)\big|^2 &= \frac{(1-|a|^2)(1-|x|^2)}{1-2x\cdot a + |x|^2|a|^2};\\
\big|T_a(x)\big| &= \frac{|x-a|}{\sqrt{1-2x\cdot a+|x|^2|a|^2}};\\
\sinh\frac{\rho(T_a(x))}{2} &= \frac{|T_a(x)|}{\sqrt{1-|T_a(x)|^2}} = \frac{|x-a|}{\sqrt{(1-|a|^2)(1-|x|^2)}};\\
\cosh\frac{\rho(T_a(x))}{2} &= \frac{1}{\sqrt{1-|T_a(x)|^2}} = \frac{\sqrt{1-2x\cdot a+|x|^2|a|^2}}{\sqrt{(1-|a|^2)(1-|x|^2)}}.
\end{aligned} \tag{2.2}$$

Using the Möbius transformations, we can define the distance from x to y in $\mathbb{B}^n$ as follows

$$\rho(x,y) = \rho\big(T_x(y)\big) = \rho\big(T_y(x)\big) = \log\frac{1+|T_y(x)|}{1-|T_y(x)|}.$$

Also using the Möbius transformations, we can define the convolution of measurable functions f and g on $\mathbb{B}^n$ by (see e.g., [40])

$$(f*g)(x) = \int_{\mathbb{B}^n} f(y)g\big(T_x(y)\big)dV(y) \tag{2.3}$$

provided this integral exists. It is easy to check that

$$f*g = g*f.$$

Furthermore, if g is radial, i.e., $g = g(\rho)$, then (see e.g., [40, Proposition 3.15])

$$(f * g) * h = f * (g * h) \tag{2.4}$$

provided $f, g, h \in L^1(\mathbb{B}^n)$.

2.4. Fourier transform on hyperbolic spaces. In this subsection we recall some basics of Fourier analysis on hyperbolic spaces and refer the reader to [20, 30, 31, 35, 56] for more information about Fourier analysis on Riemannian symmetric spaces of noncompact type.

Set

$$e_{\lambda,\zeta}(x) = \left(\frac{\sqrt{1-|x|^2}}{|x-\zeta|} \right)^{n-1+i\lambda}, \quad x \in \mathbb{B}^n,\ \lambda \in \mathbb{R},\ \zeta \in \mathbb{S}^{n-1}.$$

The Fourier transform of a function f on $\mathbb{B}^n$ can be defined as

$$\widehat{f}(\lambda, \zeta) = \int_{\mathbb{B}^n} f(x) e_{-\lambda,\zeta}(x) dV$$

provided this integral exists. If $g \in C_0^\infty(\mathbb{B}^n)$ is radial, then

$$\widehat{(f * g)} = \widehat{f} \cdot \widehat{g}.$$

Moreover, the following inversion formula holds for $f \in C_0^\infty(\mathbb{B}^n)$ (see e.g., [40]):

$$f(x) = D_n \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \zeta) e_{\lambda,\zeta}(x) |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\varsigma),$$

where $D_n = \frac{1}{2^{3-n}\pi|\mathbb{S}^{n-1}|}$ and $\mathfrak{c}(\lambda)$ is the Harish-Chandra $\mathfrak{c}$-function given by (see e.g., [40])

$$\mathfrak{c}(\lambda) = \frac{2^{n-1-i\lambda}\Gamma(n/2)\Gamma(i\lambda)}{\Gamma(\frac{n-1+i\lambda}{2})\Gamma(\frac{1+i\lambda}{2})}.$$

Similarly, there holds the Plancherel formula:

$$\int_{\mathbb{B}^n} |f(x)|^2 dV = D_n \int_{-\infty}^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda, \zeta)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\varsigma). \tag{2.5}$$

Since $e_{\lambda,\zeta}(x)$ is an eigenfunction of $\Delta_{\mathbb{H}}$ with eigenvalue $-\frac{(n-1)^2+\lambda^2}{4}$, it is easy to check that, for $f \in C_0^\infty(\mathbb{B}^n)$,

$$\widehat{\Delta_{\mathbb{H}} f}(\lambda, \zeta) = -\frac{(n-1)^2+\lambda^2}{4} \widehat{f}(\lambda, \zeta).$$

Therefore, in analogy with the Euclidean setting, we define the fractional Laplacian on hyperbolic space as follows:

$$\widehat{(-\Delta_{\mathbb{H}})^{\gamma} f}(\lambda,\zeta) = \left(\frac{(n-1)^2+\lambda^2}{4}\right)^{\gamma} \widehat{f}(\lambda,\zeta), \quad \gamma \in \mathbb{R}. \tag{2.6}$$

For more information about fractional Laplacian on hyperbolic space, we refer to [2, 4].

3. Sharp estimates of Green's functions.

In what follows, $a \lesssim b$ will stand for $a \leq Cb$ and $a \sim b$ will stand for $C^{-1}b \leq a \leq Cb$ with a positive constant C.

Let $n \geq 2$. Denote by $e^{t\Delta_{\mathbb{H}}}$ the heat kernel on $\mathbb{B}^n$. It is well known that $e^{t\Delta_{\mathbb{H}}}$ depends only on t and $\rho(x,y)$. In fact, $e^{t\Delta_{\mathbb{H}}}$ is given explicitly by the following formulas (see e.g., [15, 25]):

- If $n = 2m$, then

$$e^{t\Delta_{\mathbb{H}}} = (2\pi)^{-\frac{n+1}{2}} t^{-\frac{1}{2}} e^{-\frac{(n-1)^2}{4}t} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh\rho}} \left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^m e^{-\frac{r^2}{4t}} dr; \tag{3.1}$$

- If $n = 2m+1$, then

$$e^{t\Delta_{\mathbb{H}}} = 2^{-m-1}\pi^{-m-1/2} t^{-\frac{1}{2}} e^{-\frac{(n-1)^2}{4}t} \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m e^{-\frac{\rho^2}{4t}}. \tag{3.2}$$

An explicit expression of Green's function $(-\Delta_{\mathbb{H}}+\lambda)^{-1}$ with $\lambda > -\frac{(n-1)^2}{4}$ is given by (see [36, 50])

$$(\lambda-\Delta_{\mathbb{H}})^{-1} = (2\pi)^{-\frac{n}{2}} (\sinh\rho)^{-\frac{n-2}{2}} e^{-\frac{(n-2)\pi}{2}i} Q^{\frac{n-2}{2}}_{\theta_n(\lambda)}(\cosh\rho), \quad n \geq 3, \tag{3.3}$$

where

$$\theta_n(\lambda) = \sqrt{\lambda + \frac{(n-1)^2}{4}} - \frac{1}{2}$$

and $Q^{\frac{n-2}{2}}_{\theta_n(\lambda)}(\cosh\rho)$ is the Legendre function of second type defined by (see [16, p. 155])

$$Q^{\mu}_{\nu}(z) = e^{i(\pi\mu)} 2^{-\nu-1} \frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)} (z^2-1)^{-\mu/2} \int_0^{\pi} (z+\cos t)^{\mu-\nu-1} (\sin t)^{2\nu+1} dt, \quad \mathrm{Re}\,\nu > -1, \quad \mathrm{Re}(\nu+\mu+1) > 0. \tag{3.4}$$

Therefore, for $n \geq 3$,

$$(\lambda-\Delta_{\mathbb{H}})^{-1} = \frac{A_n}{(\sinh\rho)^{n-2}} \int_0^{\pi} (\cosh\rho + \cos t)^{\frac{n-4}{2}-\theta_n(\lambda)} (\sin t)^{2\theta_n(\lambda)+1} dt, \tag{3.5}$$

where

$$A_n = (2\pi)^{-\frac{n}{2}} \frac{\Gamma(\frac{n}{2} + \theta_n(\lambda))}{2^{\theta_n(\lambda)+1}\Gamma(\theta_n(\lambda)+1)}.$$

LEMMA 3.1. *Let $n \geq 3$. There holds, for $\lambda > -\frac{(n-1)^2}{4}$ and $\rho > 0$,*

$$(\lambda - \Delta_{\mathbb{H}})^{-1} \lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2} \left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{1+2\sqrt{\lambda+\frac{(n-1)^2}{4}}}. \tag{3.6}$$

Proof. Set, for $0 \leq \alpha < 1$,

$$f(\alpha) = \int_0^\pi (1+\alpha\cos t)^{\frac{n-4}{2}-\theta_n(\lambda)}(\sin t)^{2\theta_n(\lambda)+1}dt.$$

Then f is a continuous function on $[0,1)$ and

$$\begin{aligned}
\lim_{\alpha\to 1-} f(\alpha) &= \int_0^\pi (1+\cos t)^{\frac{n-4}{2}-\theta_n(\lambda)}(\sin t)^{2\theta_n(\lambda)+1}dt \\
&= \int_0^\pi \left(2\cos^2\frac{t}{2}\right)^{\frac{n-4}{2}-\theta_n(\lambda)} \left(2\sin\frac{t}{2}\cos\frac{t}{2}\right)^{2\theta_n(\lambda)+1} dt \\
&= 2^{\frac{n-2}{2}+\theta_n(\lambda)} \int_0^\pi \left(\cos\frac{t}{2}\right)^{n-3} \left(\sin\frac{t}{2}\right)^{2\theta_n(\lambda)+1} dt \\
&= 2^{\frac{n-2}{2}+\theta_n(\lambda)} \int_0^\pi \left(\cos\frac{t}{2}\right)^{n-3} \left(\sin\frac{t}{2}\right)^{2\sqrt{\lambda+\frac{(n-1)^2}{4}}} dt \\
&\leq 2^{\frac{n-2}{2}+\theta_n(\lambda)}\pi.
\end{aligned}$$

Therefore, there exists a constant $C > 0$ such that for each $\alpha \in [0,1)$, $|f(\alpha)| = f(\alpha) \leq C$. Thus, by (3.5),

$$\begin{aligned}
(\lambda - \Delta_{\mathbb{H}})^{-1} &= A_n \frac{(\cosh\rho)^{\frac{n-4}{2}-\theta_n(\lambda)}}{(\sinh\rho)^{n-2}} f\left(\frac{1}{\cosh\rho}\right) \\
&\lesssim \frac{(\cosh\rho)^{\frac{n-4}{2}-\theta_n(\lambda)}}{(\sinh\rho)^{n-2}} = \frac{(\cosh\rho)^{\frac{n-4}{2}-\theta_n(\lambda)}}{(2\sinh\frac{\rho}{2}\cosh\frac{\rho}{2})^{n-2}} \\
&\sim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2} \left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{2+2\theta_n(\lambda)} \\
&= \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2} \left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{1+2\sqrt{\lambda+\frac{(n-1)^2}{4}}}.
\end{aligned}$$

To get the last inequality, we use the fact $\cosh\rho \sim \cosh^2\frac{\rho}{2}$, $\rho \geq 0$. □

Next we shall give the estimates of limiting case of Green's function, namely $\lambda = \frac{(n-1)^2}{4}$. We compute, by (3.1) and (3.2),

- If $n = 2m$, then

(3.7)
$$\begin{aligned}
&\left(-\Delta_{\mathbb{H}} - \frac{(n-1)^2}{4}\right)^{-1} \\
&= \int_0^\infty e^{t(\Delta_{\mathbb{H}} + \frac{(n-1)^2}{4})} dt \\
&= \frac{1}{2(2\pi)^{\frac{n+1}{2}}} \int_\rho^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^{m-1} \left(\frac{r}{\sinh r} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{r^2}{4t}} dt\right) dr \\
&= \frac{\sqrt{\pi}}{(2\pi)^{\frac{n+1}{2}}} \int_\rho^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh \rho}} \left(-\frac{1}{\sinh r} \frac{\partial}{\partial r}\right)^{m-1} \frac{1}{\sinh r} dr.
\end{aligned}$$

To get the last equation, we use the fact

$$\int_0^\infty t^{-\frac{3}{2}} e^{-\frac{r^2}{4t}} dt = \frac{2}{r} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{1}{t}} dt = \frac{2}{r} \int_0^\infty t^{-\frac{1}{2}} e^{-t} dt = \frac{2}{r} \Gamma(1/2) = \frac{2\sqrt{\pi}}{r}. \tag{3.8}$$

- If $n = 2m+1$, then

$$\begin{aligned}
&\left(-\Delta_{\mathbb{H}} - \frac{(n-1)^2}{4}\right)^{-1} \\
&= \int_0^\infty e^{t(\Delta_{\mathbb{H}} + \frac{(n-1)^2}{4})} dt \\
&= 2^{-m-2} \pi^{-m-1/2} \left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho}\right)^{m-1} \left(\frac{\rho}{\sinh \rho} \int_0^\infty t^{-\frac{3}{2}} e^{-\frac{\rho^2}{4t}} dt\right) \\
&= 2^{-m-1} \pi^{-m} \left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho}\right)^{m-1} \frac{1}{\sinh \rho}.
\end{aligned} \tag{3.9}$$

To get the last equation, we also use (3.8).

LEMMA 3.2. *Let k be a nonnegative integer. Then there exist constants $\{a_i\}_{i=0}^k$ such that*

$$\left(-\frac{1}{\sinh \rho} \frac{\partial}{\partial \rho}\right)^{2k} \frac{1}{\sinh \rho} = \sum_{i=0}^k a_i \left(\frac{1}{\sinh \rho}\right)^{2i+2k+1}. \tag{3.10}$$

Moreover, $a_0 = (2k)!$.

Proof. We shall prove by induction. It is easy to see that (3.10) is valid for $k=0$. Now suppose that equation (3.10) is valid for $k=l$, i.e.,

$$\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{2l}\frac{1}{\sinh\rho}=\sum_{i=0}^{l}a_i\left(\frac{1}{\sinh\rho}\right)^{2i+2l+1}$$

and $a_0=(2l)!$. Then

$$\begin{aligned}
&\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{2l+2}\frac{1}{\sinh\rho}\\
&=\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{2}\sum_{i=0}^{l}a_i\left(\frac{1}{\sinh\rho}\right)^{2i+2l+1}\\
&=\sum_{i=0}^{l}(2i+2l+1)a_i\left[-\left(\frac{1}{\sinh\rho}\right)^{2i+2l+3}+(2i+2l+3)\left(\frac{1}{\sinh\rho}\right)^{2i+2l+5}\cosh^2\rho\right]\\
&=\sum_{i=0}^{l}(2i+2l+1)a_i\left[-\left(\frac{1}{\sinh\rho}\right)^{2i+2l+3}+(2i+2l+3)\left(\frac{1}{\sinh\rho}\right)^{2i+2l+5}(1+\sinh^2\rho)\right]\\
&=\sum_{i=0}^{l}(2i+2l+1)a_i\left[(2i+2l+2)\left(\frac{1}{\sinh\rho}\right)^{2i+2l+3}+(2i+2l+3)\left(\frac{1}{\sinh\rho}\right)^{2i+2l+5}\right]\\
&=\sum_{i=0}^{l+1}a_i'\left(\frac{1}{\sinh\rho}\right)^{2i+2l+3},
\end{aligned}$$

where

$$\begin{aligned}
a_0'&=a_0(2l+1)(2l+2)=(2l+2)!;\\
a_i'&=(2i+2l+1)(2i+2l+2)a_i\\
&\quad+(2i+2l-1)(2i+2l+1)a_{i-1},\quad i=1,2,\ldots,l;\\
a_{l+1}'&=(4l+1)(4l+3)a_l.
\end{aligned}$$

The desired result follows. □

LEMMA 3.3. *Let m be a nonnegative integer. There holds, for $\rho>0$,*

$$\left|\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{m}\frac{1}{\sinh\rho}\right|\lesssim\left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{2m+1}\frac{1}{\cosh\frac{\rho}{2}}. \tag{3.11}$$

Proof. If m is even, namely $m = 2k$ for some nonnegative integer k, then by Lemma 3.2,

$$\begin{aligned}\left|\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m \frac{1}{\sinh\rho}\right| &= \left|\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{2k} \frac{1}{\sinh\rho}\right| \\ &\lesssim \sum_{i=0}^{k}\left(\frac{1}{\sinh\rho}\right)^{2i+2k+1} \\ &\lesssim \left(\frac{1}{\sinh\rho}\right)^{4k+1} + \left(\frac{1}{\sinh\rho}\right)^{2k+1} \\ &\sim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{4k+1}\frac{1}{\cosh\frac{\rho}{2}}\left[\left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{4k} + \frac{\sinh^{2k}\frac{\rho}{2}}{\cosh^{2k}\frac{\rho}{2}}\right] \\ &\lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{4k+1}\frac{1}{\cosh\frac{\rho}{2}}.\end{aligned}$$

If m is odd, namely $m = 2k+1$ for some nonnegative integer k. Also by Lemma 3.2,

(3.12)
$$\begin{aligned}\left|\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m \frac{1}{\sinh\rho}\right| &= \left|\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{2k+1} \frac{1}{\sinh\rho}\right| \\ &= \left|\sum_{i=0}^{k} a_i(2i+2k+1)\left(\frac{1}{\sinh\rho}\right)^{2i+2k+3}\cosh\rho\right| \\ &\lesssim \sum_{i=0}^{k}\left(\frac{1}{\sinh\rho}\right)^{2i+2k+3}\cosh\rho \\ &\lesssim \left(\frac{1}{\sinh\rho}\right)^{4k+3}\cosh\rho + \left(\frac{1}{\sinh\rho}\right)^{2k+3}\cosh\rho \\ &\sim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{4k+3}\frac{1}{\cosh\frac{\rho}{2}}\left[\frac{\cosh\rho}{\cosh^{4k+2}\frac{\rho}{2}} + \frac{\sinh^{2k}\frac{\rho}{2}}{\cosh^{2k+2}\frac{\rho}{2}}\cosh\rho\right].\end{aligned}$$

Notice that $\cosh\rho \sim \cosh^2\frac{\rho}{2}$, $\rho \geq 0$. We have, by (3.12),

$$\left|\left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^m \frac{1}{\sinh\rho}\right| \lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{4k+3}\frac{1}{\cosh\frac{\rho}{2}}.$$

These complete the proof. □

LEMMA 3.4. *Let $n \geq 3$. There holds, for $\rho > 0$,*

$$\left(-\Delta_{\mathbb{H}} - \frac{(n-1)^2}{4}\right)^{-1} \lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2} \frac{1}{\cosh\frac{\rho}{2}}. \tag{3.13}$$

Proof. If n is even, namely $n = 2m$ for some positive integer $m \geq 2$. Then by (3.7) and Lemma 3.3,

$$\begin{aligned}\left(-\Delta_{\mathbb{H}} - \frac{(n-1)^2}{4}\right)^{-1} &\lesssim \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh\rho}} \left(\frac{1}{\sinh\frac{r}{2}}\right)^{2m-1} \frac{1}{\cosh\frac{r}{2}} dr \\ &\leq \frac{1}{\cosh\frac{\rho}{2}} \int_{\rho}^{+\infty} \frac{\sinh r}{\sqrt{\cosh r - \cosh\rho}} \left(\frac{1}{\sinh\frac{r}{2}}\right)^{2m-1} dr.\end{aligned}$$

Using the substitution $t = \sqrt{\cosh r - \cosh\rho}/\sqrt{2} = \sqrt{\sinh^2\frac{r}{2} - \sinh^2\frac{\rho}{2}}$, we have

$$\begin{aligned}\left(-\Delta_{\mathbb{H}} - \frac{(n-1)^2}{4}\right)^{-1} &\lesssim \frac{1}{\cosh\frac{\rho}{2}} \int_0^{+\infty} \left(\frac{1}{t^2 + \sinh^2\frac{\rho}{2}}\right)^{\frac{2m-1}{2}} dt \\ &= \frac{1}{\cosh\frac{\rho}{2}} \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{2m-2} \int_0^{+\infty} \left(\frac{1}{t^2+1}\right)^{\frac{2m-1}{2}} dt \\ &\sim \frac{1}{\cosh\frac{\rho}{2}} \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2}.\end{aligned}$$

If n is odd, namely $n = 2m+1$ for some positive integer $m \geq 1$. Then by (3.9) and Lemma 3.3,

$$\left(-\Delta_{\mathbb{H}} - \frac{(n-1)^2}{4}\right)^{-1} \sim \left(-\frac{1}{\sinh\rho}\frac{\partial}{\partial\rho}\right)^{m-1} \frac{1}{\sinh\rho} \lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2} \frac{1}{\cosh\frac{\rho}{2}}.$$

Thus, we have completed the proof of the lemma. □

4. Proofs of Theorem 1.1: Sharp Poincaré-Sobolev inequalities for Paneitz operators. The main purpose of this section is to establish the Sharp Poincaré-Sobolev inequalities for Paneitz operators on hyperbolic spaces, namely Theorem 1.1.

For the sake of completeness, we first provide a slightly different proof of the Hardy-Littlewood-Sobolev inequality on $\mathbb{B}^n$, which was first proved by Beckner on hyperbolic upper half spaces [8].

Indeed, as pointed out to us by Beckner [9], the HLS inequality on hyperbolic $\mathbb{B}^n$ is equivalent to the HLS inequality on the hyperbolic upper half spaces.

THEOREM 4.1. *Let* $0 < \lambda < n$ *and* $p = \frac{2n}{2n-\lambda}$. *Then for* $f, g \in L^p(\mathbb{B}^n)$,

$$\left| \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \frac{f(x)g(y)}{\left(2\sinh \frac{\rho(T_y(x))}{2}\right)^{\lambda}} dV_x dV_y \right| \leq C_{n,\lambda} \|f\|_p \|g\|_p, \tag{4.1}$$

where

$$C_{n,\lambda} = \pi^{\lambda/2} \frac{\Gamma(n/2 - \lambda/2)}{\Gamma(n - \lambda/2)} \left(\frac{\Gamma(n/2)}{\Gamma(n)} \right)^{-1+\lambda/n} \tag{4.2}$$

is the best constant for the classical Hardy-Littlewood-Sobolev constant on $\mathbb{R}^n$. *Furthermore, the constant* $C_{n,\lambda}$ *is sharp for the inequality (4.1) and there is no nonzero extremal function for the inequality (4.1).*

Proof. We have, by (2.2),

$$\begin{aligned}
&\int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \frac{f(x)g(y)}{\left(2\sinh \frac{\rho(T_y(x))}{2}\right)^{\lambda}} dV_x dV_y \\
&= \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} f(x) \left(\frac{2}{1-|x|^2} \right)^n \left(\frac{2|x-a|}{\sqrt{(1-|a|^2)(1-|x|^2)}} \right)^{-\lambda} g(y) \left(\frac{2}{1-|y|^2} \right)^n dxdy \\
&= \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} f(x) \left(\frac{2}{1-|x|^2} \right)^{n-\frac{\lambda}{2}} |x-y|^{-\lambda} g(y) \left(\frac{2}{1-|y|^2} \right)^{n-\frac{\lambda}{2}} dxdy.
\end{aligned}$$

Set $\widetilde{f} = f(x) \left(\frac{2}{1-|x|^2} \right)^{n-\frac{\lambda}{2}}$ and $\widetilde{g} = g(y) \left(\frac{2}{1-|y|^2} \right)^{n-\frac{\lambda}{2}}$. Then by the Hardy-Littlewood-Sobolev inequality on $\mathbb{R}^n$ (see Lieb [39]),

$$\begin{aligned}
&\left| \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \frac{f(x)g(y)}{\left(2\sinh \frac{\rho(T_y(x))}{2}\right)^{\lambda}} dV_x dV_y \right| \\
&\quad = \left| \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \widetilde{f}(x) |x-y|^{-\lambda} \widetilde{g}(y) dxdy \right| \\
&\quad \leq \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} |\widetilde{f}(x)| \cdot |x-y|^{-\lambda} \cdot |\widetilde{g}(y)| dxdy \\
&\quad \leq C_{n,\lambda} \left(\int_{\mathbb{B}^n} |\widetilde{f}(x)|^{\frac{2n}{2n-\lambda}} dx \right)^{\frac{2n-\lambda}{2n}} \cdot \left(\int_{\mathbb{B}^n} |\widetilde{g}(y)|^{\frac{2n}{2n-\lambda}} dy \right)^{\frac{2n-\lambda}{2n}} \\
&\quad = C_{n,\lambda} \left(\int_{\mathbb{B}^n} |f|^{\frac{2n}{2n-\lambda}} dV \right)^{\frac{2n-\lambda}{2n}} \cdot \left(\int_{\mathbb{B}^n} |g(y)|^{\frac{2n}{2n-\lambda}} dV \right)^{\frac{2n-\lambda}{2n}}.
\end{aligned}$$

Furthermore, $C_{n,\lambda}$ is sharp and there is no nonzero extreme function. The proof of Theorem 4.1 is thereby completed. □

Before the proof of Theorem 1.1, we need the following lemma.

LEMMA 4.2. *Let* $0<\alpha<n$, $0<\beta<n$ *and* $0<\alpha+\beta<n$. *Then*

$$\begin{aligned}&\int_{\mathbb{B}^n}\left(\sinh\frac{\rho(x,y)}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho(x,y)}{2}\right)^{-\alpha-\beta}\left(\sinh\frac{\rho(x,z)}{2}\right)^{\beta-n}dV_x\\&\quad=\left\{\left[\left(\sinh\frac{\rho}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho}{2}\right)^{-\alpha-\beta}\right]*\left(\sinh\frac{\rho}{2}\right)^{\beta-n}\right\}(T_z(y))\\&\quad\le 2^n\frac{\gamma_n(\alpha)\gamma_n(\beta)}{\gamma_n(\alpha+\beta)}\left(\sinh\frac{\rho(y,z)}{2}\right)^{\alpha+\beta-n}\left(\cosh\frac{\rho(y,z)}{2}\right)^{-\alpha},\end{aligned}$$

where

$$\gamma_n(\alpha)=\pi^{n/2}2^{\alpha}\frac{\Gamma(\alpha/2)}{\Gamma(\frac{n-\alpha}{2})},\quad 0<\alpha<n. \tag{4.3}$$

Proof. We firstly show

$$\begin{aligned}&\int_{\mathbb{B}^n}\left(\sinh\frac{\rho(x,y)}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho(x,y)}{2}\right)^{-\alpha-\beta}\left(\sinh\frac{\rho(x,z)}{2}\right)^{\beta-n}dV_x\\&\quad=\left\{\left[\left(\sinh\frac{\rho}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho}{2}\right)^{-\alpha-\beta}\right]*\left(\sinh\frac{\rho}{2}\right)^{\beta-n}\right\}(T_z(y)).\end{aligned} \tag{4.4}$$

In fact, using the following identity (see [40], (3.13)),

$$|T_z(T_y(x))|=|T_x(T_y(z))|,\ \ x,y,z\in\mathbb{B}^n,$$

we have, by the Möbius shift invariance,

$$\begin{aligned}&\int_{\mathbb{B}^n}\left(\sinh\frac{\rho(x,y)}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho(x,y)}{2}\right)^{-\alpha-\beta}\left(\sinh\frac{\rho(x,z)}{2}\right)^{\beta-n}dV_x\\&=\int_{\mathbb{B}^n}\left(\sinh\frac{\rho(T_y(x),y)}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho(T_y(x),y)}{2}\right)^{-\alpha-\beta}\left(\sinh\frac{\rho(T_y(x),z)}{2}\right)^{\beta-n}dV_x\\&=\int_{\mathbb{B}^n}\left(\sinh\frac{\rho(T_y(T_y(x)))}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho(T_y(T_y(x)))}{2}\right)^{-\alpha-\beta}\left(\sinh\frac{\rho(T_z(T_y(x)))}{2}\right)^{\beta-n}dV_x\\&=\int_{\mathbb{B}^n}\left(\sinh\frac{\rho(x)}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho(x)}{2}\right)^{-\alpha-\beta}\left(\sinh\frac{\rho(T_x(T_z(y)))}{2}\right)^{\beta-n}dV_x\\&=\int_{\mathbb{B}^n}\left(\sinh\frac{\rho(x)}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho(x)}{2}\right)^{-\alpha-\beta}\left(\sinh\frac{\rho(T_{T_z(y)}(x))}{2}\right)^{\beta-n}dV_x\\&=\left\{\left[\left(\sinh\frac{\rho}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho}{2}\right)^{-\alpha-\beta}\right]*\left(\sinh\frac{\rho}{2}\right)^{\beta-n}\right\}(T_z(y)).\end{aligned}$$

To finish the proof, it is enough to show

$$\begin{aligned}\left\{\left[\left(\sinh\frac{\rho}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho}{2}\right)^{-\alpha-\beta}\right]*\left(\sinh\frac{\rho}{2}\right)^{\beta-n}\right\}(y)\\ \leq 2^n\frac{\gamma_n(\alpha)\gamma_n(\beta)}{\gamma_n(\alpha+\beta)}\left(\sinh\frac{\rho(y)}{2}\right)^{\alpha+\beta-n}\left(\cosh\frac{\rho(y)}{2}\right)^{-\alpha}.\end{aligned} \tag{4.5}$$

Notice that, for $0<\alpha<n$, $0<\beta<n$ and $0<\alpha+\beta<n$, we have (see e.g., [55])

$$\int_{\mathbb{R}^n}|x|^{\alpha-n}|y-x|^{\beta-n}dx=\frac{\gamma_n(\alpha)\gamma_n(\beta)}{\gamma_n(\alpha+\beta)}|y|^{\alpha+\beta-n}. \tag{4.6}$$

Therefore, by (2.2) and (4.6),

$$\begin{aligned}&\left\{\left[\left(\sinh\frac{\rho}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho}{2}\right)^{-\alpha-\beta}\right]*\left(\sinh\frac{\rho}{2}\right)^{\beta-n}\right\}(y)\\ &=\int_{\mathbb{B}^n}\left(\sinh\frac{\rho(x)}{2}\right)^{\alpha-n}\left(\cosh\frac{\rho(x)}{2}\right)^{-\alpha-\beta}\left(\sinh\frac{\rho(T_y(x))}{2}\right)^{\beta-n}\left(\frac{2}{1-|x|^2}\right)^n dx\\ &=\int_{\mathbb{B}^n}\left(\frac{|x|}{\sqrt{1-|x|^2}}\right)^{\alpha-n}\left(\frac{1}{\sqrt{1-|x|^2}}\right)^{-\alpha-\beta}\left(\frac{|x-y|}{\sqrt{(1-|y|^2)(1-|x|^2)}}\right)^{\beta-n}\left(\frac{2}{1-|x|^2}\right)^n dx\\ &=\frac{2^n}{(1-|y|^2)^{(\beta-n)/2}}\int_{\mathbb{B}^n}|x|^{\alpha-n}|x-y|^{\beta-n}dx\\ &\leq\frac{2^n}{(1-|y|^2)^{(\beta-n)/2}}\int_{\mathbb{R}^n}|x|^{\alpha-n}|x-y|^{\beta-n}dx\\ &=\frac{2^n}{(1-|y|^2)^{(\beta-n)/2}}\cdot\frac{\gamma_n(\alpha)\gamma_n(\beta)}{\gamma_n(\alpha+\beta)}|y|^{\alpha+\beta-n}\\ &=2^n\frac{\gamma_n(\alpha)\gamma_n(\beta)}{\gamma_n(\alpha+\beta)}\left(\sinh\frac{\rho(y)}{2}\right)^{\alpha+\beta-n}\left(\cosh\frac{\rho(y)}{2}\right)^{-\alpha}.\end{aligned}$$

Thus, the desired result follows. □

Next, we will estimate the kernel function $Q_k^{-1}(\rho)$ for the $2k$-th order operator Q_k.

LEMMA 4.3. *Let k be a positive integer with $2\leq k<\frac{n}{2}$. The kernel $Q_k^{-1}(\rho)$ satisfies*

$$Q_k^{-1}(\rho)\lesssim\left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2k},\quad \rho>0. \tag{4.7}$$

Proof. Recall that

$$Q_k=\left(-\Delta_{\mathbb{H}}-\frac{(n-1)^2}{4}\right)(P_1+2)\cdot\cdots\cdot\big(P_1+k(k-1)\big).$$

We have, by (2.4),

$$Q_k^{-1} = \left(-\Delta_{\mathbb{H}} - \frac{(n-1)^2}{4}\right)^{-1} * (P_1+2)^{-1} * \cdots * (P_1 + k(k-1))^{-1},$$

where, by Lemma 3.1, $(P_1 + i(i-1))^{-1}(2 \le i \le k)$ satisfies

$$\begin{aligned}(P_1 + i(i-1))^{-1} &= \left(i(i-1) - \frac{n(n-2)}{4} - \Delta_{\mathbb{H}}\right)^{-1} \\ &= \left((i-1/2)^2 - \frac{(n-1)^2}{4} - \Delta_{\mathbb{H}}\right)^{-1} \\ &\lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2}\left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{2i}.\end{aligned} \tag{4.8}$$

We shall prove (4.7) by induction. For $k=2$, we have, by (4.8) and Lemma 4.2,

$$\begin{aligned}Q_2^{-1} &= \left(-\Delta_{\mathbb{H}} - \frac{(n-1)^2}{4}\right)^{-1} * (P_1+2)^{-1} \\ &\lesssim \left[\left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2}\frac{1}{\cosh\frac{\rho}{2}}\right] * \left[\left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2}\left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{4}\right] \\ &\le \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2} * \left[\left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2}\left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{4}\right] \\ &\lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-4}\left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{2} \\ &\le \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-4}.\end{aligned}$$

Now suppose that equation (4.8) is valid for $k=l$, i.e., $Q_l^{-1}(\rho) \lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2l}$. Then by (4.8) and Lemma 4.2,

$$\begin{aligned}Q_{l+1}^{-1}(\rho) &= Q_l^{-1}(\rho) * \big(P_1 + (l+1)l\big) \\ &\lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2l} * \left[\left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2}\left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{2l+2}\right] \\ &\lesssim \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2l-2}\left(\frac{1}{\cosh\frac{\rho}{2}}\right)^{2} \\ &\le \left(\frac{1}{\sinh\frac{\rho}{2}}\right)^{n-2l-2}.\end{aligned}$$

Thus, the proof of the lemma is completed. □

Proof of Theorem 1.1. We first prove, for some positive constant $C > 0$,

$$\int_{\mathbb{B}^n} (Q_k u) u dV \geq C \left(\int_{\mathbb{B}^n} |u|^{\frac{2n}{n-2k}} dV \right)^{\frac{n-2k}{n}}, \quad u \in C_0^\infty(\mathbb{B}^n). \tag{4.9}$$

By Lemma 4.3 and Theorem 4.1, we have

$$\begin{aligned} \int_{\mathbb{B}^n} |(Q_k^{-\frac{1}{2}} u)(x)|^2 dV &= \int_{\mathbb{B}^n} u(x)(Q_k^{-1} u)(x) dV \\ &\lesssim \int_{\mathbb{B}^n} \int_{\mathbb{B}^n} \frac{u(x)u(y)}{\left(2\sinh \frac{\rho(T_y(x))}{2}\right)^{n-2k}} dV_x dV_y \\ &\lesssim \left(\int_{\mathbb{B}^n} |u(x)|^{\frac{2n}{n+2k}} dV \right)^{\frac{n+2k}{n}}. \end{aligned} \tag{4.10}$$

On the other hand,

$$\begin{aligned} \left| \int_{\mathbb{B}^n} u(x) g(x) dV \right|^2 &= \left| \int_{\mathbb{B}^n} (Q_k^{\frac{1}{2}} u)(x)(Q_k^{-\frac{1}{2}} g)(x) dV \right|^2 \\ &\leq \int_{\mathbb{B}^n} |(Q_k^{\frac{1}{2}} u)(x)|^2 dV \cdot \int_{\mathbb{B}^n} |(Q_k^{-\frac{1}{2}} g)(x)|^2 dV. \end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11) yields

$$\begin{aligned} \left| \int_{\mathbb{B}^n} u(x) g(x) dV \right|^2 &\lesssim \int_{\mathbb{B}^n} |(Q_k^{\frac{1}{2}} u)(x)|^2 dV \left(\int_{\mathbb{B}^n} |g(x)|^{\frac{2n}{n+2k}} dV \right)^{\frac{n+2k}{n}} \\ &= \int_{\mathbb{B}^n} Q_k u(x) \cdot u(x) dV \left(\int_{\mathbb{B}^n} |g(x)|^{\frac{2n}{n+2k}} dV \right)^{\frac{n+2k}{n}}. \end{aligned} \tag{4.12}$$

Taking $g = |u|^{\frac{n+2k}{n-2k}}$, we have, by (4.12),

$$\left(\int_{\mathbb{B}^n} |u(x)|^{\frac{2n}{n-2k}} dV \right)^2 \lesssim \int_{\mathbb{B}^n} Q_k u(x) \cdot u(x) dV \left(\int_{\mathbb{B}^n} |u(x)|^{\frac{2n}{n-2k}} dV \right)^{\frac{n+2k}{n}}. \tag{4.13}$$

Therefore,

$$\left(\int_{\mathbb{B}^n} |u(x)|^{\frac{2n}{n-2k}} dV \right)^{\frac{n-2k}{n}} \lesssim \int_{\mathbb{B}^n} Q_k u(x) \cdot u(x) dV. \tag{4.14}$$

Now we shall prove inequality (1.9). Notice that, by Plancherel formula, (2.6) and (1.3),

$$
\begin{aligned}
&\int_{\mathbb{B}^n} Q_k u(x)\cdot u(x)dV\\
&= D_n \int_{-\infty}^{+\infty}\int_{\mathbb{S}^{n-1}} \frac{\lambda^2}{4}\prod_{i=2}^{k}\frac{(2i-1)^2+\lambda^2}{4}|\widehat{u}(\lambda,\zeta)|^2|\mathfrak{c}(\lambda)|^{-2}d\lambda d\sigma(\varsigma)\\
&\geq \prod_{i=2}^{k}\frac{(2i-1)^2}{4} D_n \int_{-\infty}^{+\infty}\int_{\mathbb{S}^{n-1}} \frac{\lambda^2}{4}|\widehat{u}(\lambda,\zeta)|^2|\mathfrak{c}(\lambda)|^{-2}d\lambda d\sigma(\varsigma)\\
&= \prod_{i=2}^{k}\frac{(2i-1)^2}{4}\left(\int_{\mathbb{B}^n}|\nabla_{\mathbb{H}}u|^2dV - \frac{(n-1)^2}{4}\int_{\mathbb{B}^n}u^2dV\right)\\
&\geq C\left(\int_{\mathbb{B}^n}|u|^p dV\right)^{\frac{2}{p}},\ \ 2<p\leq \frac{2n}{n-2}.
\end{aligned}
\tag{4.15}
$$

Therefore, for $2 < p \leq \frac{2n}{n-2k}$, choose $\widetilde{p} \in (2, \frac{2n}{n-2}]$ such that $\widetilde{p} < p$. By Hölder inequality and (4.15),

$$
\begin{aligned}
\int_{\mathbb{B}^n}|u|^p dV &= \int_{\mathbb{B}^n}|u|^s|u|^t dV\\
&\leq \left(\int_{\mathbb{B}^n}|u|^{\widetilde{p}}dV\right)^{\frac{s}{\widetilde{p}}}\left(\int_{\mathbb{B}^n}|u|^{\frac{2n}{n-2k}}dV\right)^{\frac{(n-2k)t}{2n}}\\
&\leq \left(\int_{\mathbb{B}^n}Q_k u(x)\cdot u(x)dV\right)^{\frac{s}{2}}\left(\int_{\mathbb{B}^n}Q_k u(x)\cdot u(x)dV\right)^{\frac{t}{2}}\\
&= \left(\int_{\mathbb{B}^n}Q_k u(x)\cdot u(x)dV\right)^{\frac{p}{2}},
\end{aligned}
\tag{4.16}
$$

where $s = (1-\frac{n-2k}{2n}p)(\frac{1}{\widetilde{p}}-\frac{n-2k}{2n})^{-1}$ and $t = p-s$. Thus, the proof of Theorem 1.1 is finished. □

Having established Theorem 1.1, Theorem 1.2 follows immediately.

5. Proof of Theorem 1.3: Sharp higher order Hardy-Sobolev-Maz'ya's inequalities. It has been shown on the Poincaré ball model $\mathbb{B}^n$ by Liu (see [41, Theorem 2.3]):

(5.1)
$$
\left(\frac{1-|x|^2}{2}\right)^{k+\frac{n}{2}}(-\Delta)^k\left[\left(\frac{1-|x|^2}{2}\right)^{k-\frac{n}{2}}f\right] = P_k f,\quad f\in C_0^\infty(\mathbb{B}^n),\ k\in\mathbb{N}.
$$

In terms of the Poincaré half space model $\mathbb{H}^n$, we can establish the following:

LEMMA 5.1. *Let k be a positive integer. There holds, for each $f \in C_0^\infty(\mathbb{H}^n)$,*

$$x_1^{\frac{n}{2}+k}(-\Delta)^k(x_1^{k-\frac{n}{2}}f) = P_k f. \tag{5.2}$$

Proof. It is enough to show

$$x_1^{\frac{n}{2}+k}\Delta^k(x_1^{k-\frac{n}{2}}f) = \prod_{i=1}^{k}\left[\Delta_{\mathbb{H}} + \frac{(n-2i)(n+2i-2)}{4}\right]f. \tag{5.3}$$

We shall prove (5.3) by induction.

A simple calculation shows, for each $\alpha \in \mathbb{R}$ and $f \in C_0^\infty(\mathbb{H}^n)$, there holds

$$\begin{aligned} x_1^{\alpha+2}\Delta(x_1^{-\alpha}f) &= x_1^2\Delta f - 2\alpha x_1\frac{\partial f}{\partial x_1} + \alpha(\alpha+1)f \\ &= [\Delta_{\mathbb{H}} + \alpha(\alpha+1)]f + (n-2-2\alpha)x_1\frac{\partial f}{\partial x_1}. \end{aligned} \tag{5.4}$$

If we choose $\alpha = \frac{n-2}{2}$ in (5.4), then

$$x_1^{\frac{n}{2}+1}\Delta(x_1^{1-\frac{n}{2}}f) = \left[\Delta_{\mathbb{H}} + \frac{n(n-2)}{4}\right]f. \tag{5.5}$$

Now suppose that equation (5.3) is valid for $k = l$, i.e.,

$$x_1^{\frac{n}{2}+l}\Delta^l(x_1^{l-\frac{n}{2}}f) = \prod_{i=1}^{l}\left[\Delta_{\mathbb{H}} + \frac{(n-2i)(n+2i-2)}{4}\right]f. \tag{5.6}$$

We note it is easy to check

$$\Delta^{l+1}(x_1 g) = x_1\Delta^{l+1}g + 2(l+1)\frac{\partial}{\partial x_1}\Delta^l g, \quad g \in C_0^\infty(\mathbb{H}^n). \tag{5.7}$$

Therefore, by (5.6) and (5.7),

$$\begin{aligned} & x_1^{\frac{n}{2}+l+1}\Delta^{l+1}\big(x_1^{l+1-\frac{n}{2}}f\big) = x_1^{\frac{n}{2}+l+1}\Delta^{l+1}\big(x_1\cdot x_1^{l-\frac{n}{2}}f\big) \\ &\quad = x_1^{\frac{n}{2}+l+1}x_1\Delta^{l+1}\big(x_1^{l-\frac{n}{2}}f\big) + 2(l+1)x_1^{\frac{n}{2}+l+1}\frac{\partial}{\partial x_1}\Delta^l\big(x_1^{l-\frac{n}{2}}f\big) \\ &\quad = x_1^{\frac{n}{2}+l+2}\Delta\left\{x_1^{-\frac{n}{2}-l}\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}} + \frac{(n-2i)(n+2i-2)}{4}\right]f\right\} \\ &\qquad + 2(l+1)x_1^{\frac{n}{2}+l+1}\frac{\partial}{\partial x_1}\left\{x_1^{-\frac{n}{2}-l}\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}} + \frac{(n-2i)(n+2i-2)}{4}\right]f\right\}. \end{aligned} \tag{5.8}$$

By (5.4), we have

$$\begin{aligned} & x_1^{\frac{n}{2}+l+2}\Delta\left\{x_1^{-\frac{n}{2}-l}\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f\right\} \\ &= \left[\Delta_{\mathbb{H}}+\frac{(n+2l)(n+2l+2)}{4}\right]\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f \\ &\quad -2(l+2)x_1\frac{\partial}{\partial x_1}\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f. \end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9) yields

$$\begin{aligned} & x_1^{\frac{n}{2}+l+1}\Delta^{l+1}\left(x_1^{l+1-\frac{n}{2}}f\right) \\ &= \left[\Delta_{\mathbb{H}}+\frac{(n+2l)(n+2l+2)}{4}\right]\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f \\ &\quad -2(l+2)x_1\frac{\partial}{\partial x_1}\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f \\ &\quad +2(l+1)x_1^{\frac{n}{2}+l+1}\frac{\partial}{\partial x_1}\left\{x_1^{-\frac{n}{2}-l}\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f\right\} \\ &= \left[\Delta_{\mathbb{H}}+\frac{(n+2l)(n+2l+2)}{4}\right]\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f \\ &\quad -2(l+1)x_1^{1-\frac{n}{2}-l}\cdot\frac{\partial x_1^{\frac{n}{2}+l}}{\partial x_1}\cdot\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f. \end{aligned} \tag{5.10}$$

To get the last equation in the above, we use the fact

$$\begin{aligned} & x_1\frac{\partial}{\partial x_1}\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f \\ &= x_1\frac{\partial}{\partial x_1}\left\{x_1^{\frac{n}{2}+l}x_1^{-\frac{n}{2}-l}\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f\right\} \\ &= x_1^{\frac{n}{2}+l+1}\frac{\partial}{\partial x_1}\left\{x_1^{-\frac{n}{2}-l}\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f\right\} \\ &\quad +x_1^{1-\frac{n}{2}-l}\frac{\partial x_1^{\frac{n}{2}+l}}{\partial x_1}\cdot\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f. \end{aligned}$$

Therefore, by (5.10),

$$
\begin{aligned}
&x_1^{\frac{n}{2}+l+1}\Delta^{l+1}\left(x_1^{l+1-\frac{n}{2}}f\right)\\
&\quad=\left[\Delta_{\mathbb{H}}+\frac{(n+2l)(n+2l+2)}{4}\right]\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f\\
&\qquad-2(l+1)\left(\frac{n}{2}+l\right)\prod_{i=1}^{l}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f\\
&\quad=\prod_{i=1}^{l+1}\left[\Delta_{\mathbb{H}}+\frac{(n-2i)(n+2i-2)}{4}\right]f.
\end{aligned}
\tag{5.11}
$$

The proof of Lemma 5.1 is thus completed. □

Proof of Theorem 1.3. We claim that

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}|\nabla^k u|^2dx-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\int_{\mathbb{R}^n_+}\frac{u^2}{x_1^{2k}}dx\\
&\quad=\int_{\mathbb{B}^n}(P_kv)vdV-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\int_{\mathbb{B}^n}v^2dV,
\end{aligned}
\tag{5.12}
$$

where $v=x_1^{\frac{n}{2}-k}u$. In fact, by Lemma 5.1,

$$
\begin{aligned}
&\int_{\mathbb{R}^n_+}|\nabla^k u|^2dx-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\int_{\mathbb{R}^n_+}\frac{u^2}{x_1^{2k}}dx\\
&\quad=\int_{\mathbb{R}^n_+}u\cdot(-\Delta)^k udx-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\int_{\mathbb{R}^n_+}\frac{u^2}{x_1^{2k}}dx\\
&\quad=\int_{\mathbb{H}^n}x_1^{\frac{n}{2}-k}u\cdot P_k(x_1^{\frac{n}{2}-k}u)dV-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\int_{\mathbb{H}^n}(x_1^{\frac{n}{2}-k}u)^2dV\\
&\quad=\int_{\mathbb{B}^n}(P_kv)vdV-\prod_{i=1}^{k}\frac{(2i-1)^2}{4}\int_{\mathbb{B}^n}v^2dV.
\end{aligned}
\tag{5.13}
$$

This proves the claim. Therefore, by Theorem 1.2,

$$\begin{aligned}
&\int_{\mathbb{R}^n_+} |\nabla^k u|^2 dx - \prod_{i=1}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{R}^n_+} \frac{u^2}{x_1^{2k}} dx \\
&= \int_{\mathbb{B}^n} (P_k v) v dV - \prod_{i=1}^{k} \frac{(2i-1)^2}{4} \int_{\mathbb{B}^n} v^2 dV \\
&\geq C \left(\int_{\mathbb{H}^n} |v|^p dV \right)^{\frac{2}{p}} = C \left(\int_{\mathbb{R}^n_+} x_1^{\gamma} |u|^p dx \right)^{\frac{2}{p}}.
\end{aligned}$$

Similarly, using the identity (5.1), we obtain inequality (1.13). The proof of Theorem 1.3 is thereby completed. □

6. proof of Theorem 1.4: Fractional Poincaré-Sobolev inequality. The main purpose of this section is two fold. We first establish the Poincaré-Sobolev inequality of fractional order s with best Sobolev constant in dimensional 3. When $s = 1$, this is the result by Benguria, Frank and Loss (see [10]). Next, we will illustrate that Theorem 6.2 below does not imply our sharp Hardy-Sobolev-Maz'ya inequalities.

We first show the following lemma, which is equivalent to Theorem 1.4.

LEMMA 6.1. *Let* $\frac{s}{2} \leq s < \frac{3}{2}$. *Then for* $f \in L^{\frac{6}{3+2s}}(\mathbb{B}^3)$, *we have*

$$\begin{aligned}
&\int_{\mathbb{B}^3} f(x) \left[(-\Delta_{\mathbb{H}} - 1)^{-s} f \right](x) dV \\
&\leq 2^{-2s} \pi^{-\frac{3}{2}} \frac{\Gamma(\frac{3-2s}{2})}{\Gamma(s)} C_{3,3-2s} \left(\int_{\mathbb{B}^3} |f|^{\frac{6}{3+2s}} dV \right)^{\frac{3+2s}{3}},
\end{aligned} \tag{6.1}$$

where $C_{3,3-2s}$ *is given by (4.2). Furthermore, the constant* $C_{3,3-2s}$ *is sharp and this constant is not attained in (6.1) for nonzero functions.*

Proof. We have, for $0 < s < 3/2$,

$$\begin{aligned}
(-\Delta_{\mathbb{H}} - 1)^{-s} &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} e^{t(\Delta_{\mathbb{H}}+1)} dt \\
&= \frac{1}{\Gamma(s)} \cdot 2^{-3} \pi^{-3/2} \frac{\rho}{\sinh \rho} \int_0^{\infty} t^{\frac{2s-5}{2}} e^{-\frac{\rho^2}{4t}} dt \\
&= 2^{-\alpha} \pi^{-3/2} \frac{\Gamma(\frac{3-2s}{2})}{\Gamma(s)} \frac{1}{\rho^{2-2s} \sinh \rho} \\
&= 2^{-\alpha} \pi^{-3/2} \frac{\Gamma(\frac{3-2s}{2})}{\Gamma(s)} \left(\frac{1}{2 \sinh \frac{\rho}{2}} \right)^{3-2s} \cdot \Psi_s(\rho),
\end{aligned}$$

where $\Psi_s(\rho) = \left(\frac{2\sinh\frac{\rho}{2}}{\rho}\right)^{2-2s} \frac{1}{\cosh\frac{\rho}{2}}$. It is easy to check, for $s \geq \frac{1}{2}$, the function $\Psi_s(\rho)$ is decreasing on $(0,\infty)$ and

$$\sup_{\rho\in(0,\infty)} \Psi_s(\rho) = 1.$$

Therefore, by Theorem 4.1,

$$\begin{aligned}
&\int_{\mathbb{B}^3} f(x)\left[(-\Delta_{\mathbb{H}}-1)^{-s}f\right](x)dV \\
&\quad\leq 2^{-\alpha}\pi^{-3/2}\frac{\Gamma(\frac{3-2s}{2})}{\Gamma(s)}\int_{\mathbb{B}^3}\int_{\mathbb{B}^3}\frac{f(x)f(y)}{\left(2\sinh\frac{\rho(T_y(x))}{2}\right)^{3-2s}}dV_x dV_y \\
&\quad\leq 2^{-2s}\pi^{-\frac{3}{2}}\frac{\Gamma(\frac{3-2s}{2})}{\Gamma(s)}C_{3,3-2s}\left(\int_{\mathbb{B}^3}|f|^{\frac{6}{3+2s}}dV\right)^{\frac{3+2s}{3}}. \qquad \square
\end{aligned}$$

Next we recall a result of Benguria, Frank and Loss (see [10, Corollary 3.1]).

THEOREM 6.2. *Let $n \geq 2$ and $n-1 \leq \alpha < n$ (resp. $0<\alpha<1$ if $n=1$). The operator $(-\Delta - \frac{1}{4x_1^2})^{-\frac{\alpha}{2}}$ is a bounded operator from $L^p(\mathbb{R}^n_+)$ to $L^q(\mathbb{R}^n_+)$ for all $1<p,q<\infty$ that satisfy $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$, and its norm coincides with the one of $(-\Delta)^{-\frac{\alpha}{2}} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$.*

Moreover, for such values of α we have

$$\left(f,(-\Delta-\frac{1}{4x_1^2})^{-\frac{\alpha}{2}}f\right) \leq 2^{-\alpha}\pi^{-\frac{n}{2}}\frac{\Gamma(\frac{n-\alpha}{2})}{\Gamma(\frac{\alpha}{2})}C_{n,n-\alpha}\|f\|_p, \tag{6.2}$$

where $p=\frac{2n}{n+\alpha}$ and $C_{n,n-\alpha}$ is given by (4.2). Furthermore, the constant $C_{n,n-\alpha}$ is the sharp constant and this constant is not attained in (6.2) for nonzero functions.

Remark 6.3. We note that the Hardy-Sobolev-Maz'ya inequality for higher order derivatives proved in our paper does not follow from the above result of [10]. This can be explained through the Fourier analysis on hyperbolic spaces. For simplicity, we choosing $n=5$ and $\alpha=4$ in (6.2) to illustrate this.

Following the proof in [10], we have the following sharp Sobolev type inequality

$$\int_{\mathbb{R}^5_+}\left|\Delta u+\frac{u}{4x_1^2}\right|^2 dx \geq S_{5,2}\left(\int_{\mathbb{R}^5_+}|u|^{10}dx\right)^{\frac{1}{5}}, \tag{6.3}$$

where $S_{5,2}$ is the best second order Sobolev constant on $\mathbb{R}^5$. Inequality (6.3) is equivalent to (see Lemma 6.4)

$$\int_{\mathbb{H}^5} |\Delta_{\mathbb{H}} f + 3f|^2 dV \geq S_{5,2} \left(\int_{\mathbb{H}^5} |f|^{10} dV \right)^{\frac{1}{5}}, \ \ f = x_1^{\frac{1}{2}} u. \tag{6.4}$$

By the Plancherel formula (2.5),

$$\int_{\mathbb{H}^5} |\Delta_{\mathbb{H}} f + 3f|^2 dV = D_5 \int_{-\infty}^{+\infty} \int_{\mathbb{S}^4} \frac{(\lambda^2+4)^2}{16} |\widehat{f}(\lambda,\zeta)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\varsigma).$$

However,

$$\begin{aligned}
&\int_{\mathbb{H}^5} (P_4 f) f dV - \prod_{i=1}^{2} \frac{(2i-1)^2}{4} \int_{\mathbb{H}^5} f^2 dV \\
&= D_5 \int_{-\infty}^{+\infty} \int_{\mathbb{S}^4} \prod_{i=1}^{2} \frac{\lambda^2 + (2i-1)^2}{4} |\widehat{f}(\lambda,\zeta)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\varsigma) \\
&\quad - D_5 \prod_{i=1}^{2} \frac{(2i-1)^2}{4} \int_{-\infty}^{+\infty} \int_{\mathbb{S}^4} |\widehat{f}(\lambda,\zeta)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\varsigma) \\
&= D_5 \int_{-\infty}^{+\infty} \int_{\mathbb{S}^4} \frac{\lambda^4 + 10\lambda^2}{16} |\widehat{f}(\lambda,\zeta)|^2 |\mathfrak{c}(\lambda)|^{-2} d\lambda d\sigma(\varsigma).
\end{aligned}$$

Since

$$\inf_{\lambda \in \mathbb{R}} \frac{\frac{\lambda^4+10\lambda^2}{16}}{\frac{(\lambda^2+4)^2}{16}} = 0,$$

then one cannot find a positive constant C which is independent of f such that

$$\int_{\mathbb{H}^5} (P_4 f) f dV - \prod_{i=1}^{2} \frac{(2i-1)^2}{4} \int_{\mathbb{H}^5} f^2 dV \geq C \int_{\mathbb{H}^5} |\Delta_{\mathbb{H}} f + 3f|^2 dV.$$

Therefore, inequality (6.3) does not imply the following Hardy-Sobolev-Maz'ya inequality on $\mathbb{R}^5_+$

$$\int_{\mathbb{R}^5_+} |\Delta u|^2 dx - \frac{9}{16} \int_{\mathbb{R}^5_+} \frac{u^2}{x_1^4} dx \geq C \left(\int_{\mathbb{R}^5_+} |u|^{10} dx \right)^{\frac{1}{5}}.$$

This shows that Theorem 6.2 does not lead to the fourth order Hardy-Sobolev-Maz'ya inequality on the hyperbolic space in dimension 5.

We now complete the proof of the following lemma which was needed in the above illustration.

LEMMA 6.4. *Let* $u \in C_0^\infty(\mathbb{R}^5_+)$ *and* $f = x_1^{\frac{1}{2}} u$. *Then*

$$\int_{\mathbb{R}^5_+} \left|\Delta u + \frac{u}{4x_1^2}\right|^2 dx = \int_{\mathbb{H}^5} |\Delta_{\mathbb{H}} f + 3f|^2 dV.$$

Proof. We compute

$$\begin{aligned} \int_{\mathbb{R}^5_+} \left|\Delta u + \frac{u}{4x_1^2}\right|^2 dx &= \int_{\mathbb{R}^5_+} \left|-\Delta u - \frac{u}{4x_1^2}\right|^2 dx \\ &= \int_{\mathbb{R}^5_+} \left(u(-\Delta)^2 u - \frac{u(-\Delta)u}{2x_1^2} + \frac{u^2}{16x_1^4} \right) dx. \end{aligned} \tag{6.5}$$

Since $u = x_1^{-\frac{1}{2}} f$, we have, by (5.4) and (5.2),

$$\begin{aligned} (-\Delta)u &= (-\Delta)(x_1^{-\frac{1}{2}} f) = x_1^{-\frac{5}{2}} \left(-\Delta_{\mathbb{H}} - \frac{3}{4}\right) f - 2x_1^{-\frac{3}{2}} \frac{\partial f}{\partial x_1}; \\ (-\Delta)^2 u &= (-\Delta)^2 (x_1^{-\frac{1}{2}} f) = x_1^{-\frac{9}{2}} \left(-\Delta_{\mathbb{H}} - \frac{15}{4}\right)\left(-\Delta_{\mathbb{H}} - \frac{7}{4}\right) f. \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{\mathbb{R}^5_+} u(-\Delta)^2 u &= \int_{\mathbb{R}^5_+} x_1^{-5} f \left(-\Delta_{\mathbb{H}} - \frac{15}{4}\right)\left(-\Delta_{\mathbb{H}} - \frac{7}{4}\right) f dx \\ &= \int_{\mathbb{H}^5} f \left(-\Delta_{\mathbb{H}} - \frac{15}{4}\right)\left(-\Delta_{\mathbb{H}} - \frac{7}{4}\right) f dV; \\ -\int_{\mathbb{R}^5_+} \frac{u(-\Delta)u}{2x_1^2} dx &= -\frac{1}{2}\int_{\mathbb{R}^5_+} x_1^{-5} f \left(-\Delta_{\mathbb{H}} - \frac{3}{4}\right) f dx + \int_{\mathbb{R}^5_+} x_1^{-4} f \frac{\partial f}{\partial x_1} dx \\ &= -\frac{1}{2}\int_{\mathbb{H}^5} f \left(-\Delta_{\mathbb{H}} - \frac{3}{4}\right) f dV + 2\int_{\mathbb{R}^5_+} x_1^{-5} f^2 dx \\ &= -\frac{1}{2}\int_{\mathbb{H}^5} f \left(-\Delta_{\mathbb{H}} - \frac{3}{4}\right) f dV + 2\int_{\mathbb{H}^5} f^2 dV; \\ \int_{\mathbb{R}^5_+} \frac{u^2}{16x_1^4} dx &= \int_{\mathbb{R}^5_+} \frac{f^2}{16x_1^5} dx = \frac{1}{16}\int_{\mathbb{H}^5} f^2 dV. \end{aligned} \tag{6.6}$$

Combining (6.5) and (6.6) yields

$$\begin{aligned}\int_{\mathbb{R}^5_+}\left|-\Delta u-\frac{u}{4x_1^2}\right|^2dx &= \int_{\mathbb{H}^5} f\left(-\Delta_{\mathbb{H}}-\frac{15}{4}\right)\left(-\Delta_{\mathbb{H}}-\frac{7}{4}\right)fdV\\ &\quad -\frac{1}{2}\int_{\mathbb{H}^5} f\left(-\Delta_{\mathbb{H}}-\frac{3}{4}\right)fdV+\frac{33}{16}\int_{\mathbb{H}^5} f^2dx\\ &= \int_{\mathbb{H}^5} f\left(-\Delta_{\mathbb{H}}-3\right)^2 fdV\\ &= \int_{\mathbb{H}^5} |\Delta_{\mathbb{H}}f+3f|^2dV.\end{aligned}$$

This completes the proof. □

7. An alternative proof of the sharp Sobolev inequalities on hyperbolic spaces. In this section, we shall give an alternative proof of the sharp constant for the Sobolev inequality in hyperbolic spaces via Theorem 4.1. This Sobolev inequality without subtracting the Hardy term is weaker than the Poincaré-Sobolev and Hardy-Sobolev-Maz'ya inequalities we established in Sections 4 and 5. The sharp Sobolev inequality on n-dimensional sphere $\mathbb{S}^n$ was proved by Beckner [5]. The authors thank one of the referees who points out and insists that Beckner's theorem on the sphere can imply the main theorem on sharp Sobolev inequalities on hyperbolic spaces by Liu in [41] and thus can give a simpler proof of Liu's results [41]. In this section, we will provide a different proof of this sharp Sobolev inequality on hyperbolic spaces using the Hardy-Littlewood-Sobolev inequality.

Before we begin the proof, we need the following lemma.

LEMMA 7.1. *Let $f\in C_0^\infty(\mathbb{B}^n)$. Then there exists a function $g\in C_0^\infty(\mathbb{B}^n)$ such that $P_kg=f$.*

Proof. Choose $\widetilde{f}\in C_0^\infty(\mathbb{B}^n)$ such that

$$(-\Delta)^k\widetilde{f}=\left(\frac{1-|x|^2}{2}\right)^{-k-\frac{n}{2}}f.$$

Set $g=(\frac{1-|x|^2}{2})^{k-\frac{n}{2}}\widetilde{f}$. Then $g\in C_0^\infty(\mathbb{B}^n)$. Furthermore, by (5.1),

$$P_kg=P_k\left[\left(\frac{1-|x|^2}{2}\right)^{k-\frac{n}{2}}\widetilde{f}\right]=\left(\frac{1-|x|^2}{2}\right)^{k+\frac{n}{2}}(-\Delta)^k\widetilde{f}=f. \qquad \square$$

It is known that the kernel $(-\Delta)^{-k}(1\le k<n/2)$ is $\frac{1}{\gamma_n(2k)|x|^{n-2k}}$, where

$$\gamma_n(2k)=\frac{\pi^{n/2}2^{2k}\Gamma(k)}{\Gamma(\frac{n}{2}-k)}. \tag{7.1}$$

We have, for $f \in C_0^\infty(\mathbb{B}^n)$,

$$f(0) = \frac{1}{\gamma_n(2k)} \int_{\mathbb{B}^n} (-\Delta)^k f(x) \cdot \frac{1}{|x|^{n-2k}} dx. \tag{7.2}$$

Replacing f by $(1-|x|^2)^{k-\frac{n}{2}} f$ and using (5.1), we obtain

$$\begin{aligned} f(0) &= \frac{1}{\gamma_n(2k)} \int_{\mathbb{B}^n} (-\Delta)^k [(1-|x|^2)^{k-\frac{n}{2}} f] \cdot \frac{1}{|x|^{n-2k}} dx \\ &= \frac{2^{k-\frac{n}{2}}}{\gamma_n(2k)} \int_{\mathbb{B}^n} P_k f(x) \cdot \frac{1}{|x|^{n-2k}} \left(\frac{1-|x|^2}{2}\right)^{-\frac{n}{2}-k} dx \\ &= \frac{1}{\gamma_n(2k)} \int_{\mathbb{B}^n} P_k f(x) \cdot \left(\frac{\sqrt{1-|x|^2}}{2|x|}\right)^{n-2k} dV \\ &= \frac{1}{\gamma_n(2k)} \int_{\mathbb{B}^n} P_k f(x) \cdot \left(2\sinh\frac{\rho(x)}{2}\right)^{2k-n} dV. \end{aligned} \tag{7.3}$$

Therefore, by the Möbius shift invariance, for $f \in C_0^\infty(\mathbb{B}^n)$ and $y \in \mathbb{B}^n$,

$$f(y) = \frac{1}{\gamma_n(2k)} \int_{\mathbb{B}^n} P_k f(x) \cdot \left(2\sinh\frac{\rho(T_y(x))}{2}\right)^{2k-n} dV_x. \tag{7.4}$$

THEOREM 7.2. *Let $1 \le k < n/2$. Then, for any $f \in C_0^\infty(\mathbb{B}^n)$,*

$$\int_{\mathbb{B}^n} P_k f(x) \cdot f(x) dV \ge S_{n,k} \left(\int_{\mathbb{B}^n} |f(x)|^{\frac{2n}{n-2k}} dV\right)^{\frac{n-2k}{n}},$$

where $S_{n,k}$ is the best k-th order Sobolev constant. Furthermore, the inequality is strict for nonzero f's.

Proof. Let $g \in C_0^\infty(\mathbb{B}^n)$ be such that $P_k g = f$. Then

$$\begin{aligned} \int_{\mathbb{B}^n} u(x) g(x) dV &= \int_{\mathbb{B}^n} (P_k^{-1} f)(x) \cdot (P_k g)(x) dV \\ &= \int_{\mathbb{B}^n} (P_k^{-1} f)(x) \cdot f(x) dV \\ &= \int_{\mathbb{B}^n} |(P_k^{-1/2} f)(x)|^2 dV. \end{aligned} \tag{7.5}$$

By (7.4) and Theorem 1.2,

$$
\begin{aligned}
\int_{\mathbb{B}^n} u(x)g(x)dV &= \frac{1}{\gamma_n(2k)}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n} \frac{f(x)\cdot P_k g}{(2\sinh\frac{\rho(T_y(x))}{2})^{n-2k}}dV_x dV_y \\
&= \frac{1}{\gamma_n(2k)}\int_{\mathbb{B}^n}\int_{\mathbb{B}^n} \frac{f(x)\cdot f(y)}{(2\sinh\frac{\rho(T_y(x))}{2})^{n-2k}}dV_x dV_y \\
&\leq \frac{1}{\gamma_n(2k)}C_{n,n-2k}\left(\int_{\mathbb{B}^n}|f(x)|^{\frac{2n}{n+2k}}dV\right)^{\frac{n+2k}{n}}.
\end{aligned} \tag{7.6}
$$

Combining (7.5) and (7.6) yields

$$
\begin{aligned}
&\int_{\mathbb{B}^n}|(P_k^{-1/2}f)(x)|^2 dV \\
&\quad\leq \frac{1}{\gamma_n(2k)}C_{n,n-2k}\left(\int_{\mathbb{B}^n}|f(x)|^{\frac{2n}{n+2k}}dV\right)^{\frac{n+2k}{n}}, \quad f\in C_0^\infty(\mathbb{B}^n).
\end{aligned} \tag{7.7}
$$

On the other hand, for $h\in C_0^\infty(\mathbb{B}^n)$,

$$
\begin{aligned}
\left|\int_{\mathbb{B}^n} h(x)f(x)dV\right|^2 &= \left|\int_{\mathbb{B}^n}(P_k^{\frac{1}{2}}h)(x)(P_k^{-\frac{1}{2}}f)(x)dV\right|^2 \\
&\leq \int_{\mathbb{B}^n}|(P_k^{\frac{1}{2}}h)(x)|^2 dV\cdot\int_{\mathbb{B}^n}|(P_k^{-\frac{1}{2}}f)(x)|^2 dV.
\end{aligned} \tag{7.8}
$$

Combining (7.7) and (7.8) yields

$$
\begin{aligned}
&\left|\int_{\mathbb{B}^n} h(x)f(x)dV\right|^2 \\
&\quad\leq \frac{1}{\gamma_n(2k)}C_{n,n-2k}\int_{\mathbb{B}^n}|(P_k^{\frac{1}{2}}h)(x)|^2 dV\left(\int_{\mathbb{B}^n}|f(x)|^{\frac{2n}{n+2k}}dV\right)^{\frac{n+2k}{n}} \\
&\quad= \frac{1}{\gamma_n(2k)}C_{n,n-2k}\int_{\mathbb{B}^n}P_k h(x)\cdot h(x)dV\left(\int_{\mathbb{B}^n}|f(x)|^{\frac{2n}{n+2k}}dV\right)^{\frac{n+2k}{n}}.
\end{aligned} \tag{7.9}
$$

Taking $f=|h|^{\frac{n+2k}{n-2k}}$, we have, by (7.9),

$$
\begin{aligned}
&\left(\int_{\mathbb{B}^n}|h(x)|^{\frac{2n}{n-2k}}dV\right)^2 \\
&\quad\leq \frac{1}{\gamma_n(2k)}C_{n,n-2k}\int_{\mathbb{B}^n}P_k h(x)\cdot h(x)dV\left(\int_{\mathbb{B}^n}|h(x)|^{\frac{2n}{n-2k}}dV\right)^{\frac{n+2k}{n}}.
\end{aligned} \tag{7.10}
$$

Therefore, we have, for $h\in C_0^\infty(\mathbb{B}^n)$,

$$
\frac{\gamma_n(2k)}{C_{n,n-2k}}\left(\int_{\mathbb{B}^n}|h(x)|^{\frac{2n}{n-2k}}dV\right)^{\frac{n-2k}{n}} \leq \int_{\mathbb{B}^n}P_k h(x)\cdot h(x)dV, \tag{7.11}
$$

where

$$\frac{\gamma_n(2k)}{C_{n,n-2k}} = \frac{\pi^{n/2}2^{2k}\Gamma(k)}{\Gamma(n/2-k)} \cdot \frac{\Gamma(n/2+k)}{\pi^{n/2-k}\Gamma(k)} \left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{2k/n}$$
$$= 2^{2k}\pi^k \frac{\Gamma(n/2+k)}{n/2-k} \left(\frac{\Gamma(n/2)}{\Gamma(n)}\right)^{2k/n}$$

is the best k-th order Sobolev constant (see [39]). Moreover, by Theorem 4.1, the inequality is strict for nonzero f's. □

8. Proof of Theorem 1.5: the best constants in Hardy-Sobolev-Maz'ya and Sobolev inequalities coincide in dimension 5. In this section we shall show that the sharp constant of Hardy-Sobolev-Maz'ya inequality for $n=5$ and $k=2$ coincides with the corresponding Sobolev constant. We also give an evidence showing that such a coincidence in dimension 6 does not hold (see Remark 8.3 and Lemma 8.4. As we have pointed out, we can continue to establish that the sharp constant of Hardy-Sobolev-Maz'ya inequality for special n and $k=2$ coincides with the corresponding Sobolev constant (see remarks after the statement of Theorem 1.5 and Conjecture 1.6 in the introduction).

The proof of Theorem 1.5 depends on the following key lemma.

LEMMA 8.1. *Let $n=5$. There holds*

$$[(-4-\Delta_{\mathbb{H}})(-3-\Delta_{\mathbb{H}})]^{-1} = \frac{1}{16\pi^2} \cdot \frac{1}{2\sinh\frac{\rho}{2}} \cdot \frac{1}{\cosh^2\frac{\rho}{2}} \leq \frac{1}{\gamma_5(4)} \cdot \frac{1}{2\sinh\frac{\rho}{2}}, \qquad \rho > 0,$$

where $\gamma_5(4) = \frac{1}{16\pi^2}$ is defined in (4.3).

Proof. By (3.9), we have, for $n=5$,

$$(-\Delta_{\mathbb{H}}-4)^{-1} = \int_0^\infty e^{(-\Delta_{\mathbb{H}}-4)t}dt = \frac{1}{8\pi^2} \cdot \frac{\cosh\rho}{\sinh^3\rho}.$$

On the other hand, choosing $\nu=1$ in (3.5), we have

$$(-3-\Delta_{\mathbb{H}})^{-1} = \frac{1}{(\sinh\rho)^3} \cdot (2\pi)^{-\frac{5}{2}} \frac{\Gamma(\frac{5}{2}+\frac{1}{2})}{2^{\frac{1}{2}+1}\Gamma(\frac{1}{2}+1)} \int_0^\pi \sin^2 t dt = \frac{1}{8\pi^2} \cdot \frac{1}{\sinh^3\rho}.$$

Therefore,

$$\begin{aligned}\big[\big(-4-\Delta_{\mathbb{H}}\big)\big(-3-\Delta_{\mathbb{H}}\big)\big]^{-1} &= \big(-4-\Delta_{\mathbb{H}}\big)^{-1} - \big(-3-\Delta_{\mathbb{H}}\big)^{-1} \\ &= \frac{1}{8\pi^2} \cdot \frac{\cosh\rho}{\sinh^3\rho} - \frac{1}{8\pi^2} \cdot \frac{1}{\sinh^3\rho} \\ &= \frac{1}{16\pi^2} \cdot \frac{1}{2\sinh\frac{\rho}{2}} \cdot \frac{1}{\cosh^2\frac{\rho}{2}}\end{aligned}$$

The desired result follows. □

By Lemma 8.1 and Theorem 4.1, we have

Corollary 8.2. There holds, for each $u \in C_0^\infty(\mathbb{B}^5)$,

$$\int_{\mathbb{B}^5} ((-4-\Delta_{\mathbb{H}})(-3-\Delta_{\mathbb{H}})u)udV \geq S_{5,2}\left(\int_{\mathbb{B}^n} |u|^{10}dV\right)^{\frac{1}{5}},$$

where $S_{5,2} = \gamma_5(4)/C_{5,1}$ is the best second order Sobolev constant. The inequality is strict for any nonzero u.

Proof of Theorem 1.5. By Corollary 8.2, it is enough to show

$$\int_{\mathbb{B}^5} (P_2u)udV - \frac{9}{16}\int_{\mathbb{B}^5} u^2dV \geq \int_{\mathbb{B}^5} ((-4-\Delta_{\mathbb{H}})(-3-\Delta_{\mathbb{H}})u)udV.$$

In fact, by the Plancherel formula (see (2.5)),

$$\begin{aligned}
&\int_{\mathbb{B}^5} (P_2u)udV - \frac{9}{16}\int_{\mathbb{B}^5} u^2dV \\
&= D_5\int_{-\infty}^{+\infty}\int_{\mathbb{S}^4}\left[\frac{(\lambda^2+1)(\lambda^2+9)}{16} - \frac{9}{16}\right]|\widehat{u}(\lambda,\zeta)|^2|\mathfrak{c}(\lambda)|^{-2}d\lambda d\sigma(\varsigma) \\
&= D_5\int_{-\infty}^{+\infty}\int_{\mathbb{S}^4}\frac{\lambda^4+10\lambda^2}{16}|\widehat{u}(\lambda,\zeta)|^2|\mathfrak{c}(\lambda)|^{-2}d\lambda d\sigma(\varsigma) \\
&\geq D_5\int_{-\infty}^{+\infty}\int_{\mathbb{S}^4}\frac{\lambda^2(\lambda^2+1)}{16}|\widehat{u}(\lambda,\zeta)|^2|\mathfrak{c}(\lambda)|^{-2}d\lambda d\sigma(\varsigma) \\
&= \int_{\mathbb{B}^5} ((-4-\Delta_{\mathbb{H}})(-3-\Delta_{\mathbb{H}})u)udV.
\end{aligned}$$

This completes the proof. □

Remark 8.3. It seems that the sharp constant of Poincaré-Sobolev inequalities in case $n = 6$ and $k = 2$ is strictly less than Sobolev constant. In fact, we have the following:

LEMMA 8.4. *Let $n = 6$. There exists $\epsilon > 0$ such that for $0 < \rho < \epsilon$,*

(8.1)

$$\big[\big(-25/4-\Delta_{\mathbb{H}}\big)\big(-4-\Delta_{\mathbb{H}}\big)\big]^{-1} > \frac{1}{16\pi^3}\cdot\frac{1}{(2\sinh\frac{\rho}{2})^2} = \frac{1}{\gamma_6(4)}\cdot\frac{1}{(2\sinh\frac{\rho}{2})^2}.$$

Before the proof, we need some fact about the Legendre function of second type $Q_\nu^\mu(z)$. It is known that (see [23, p. 773, 7.133(2)])

$$\int_u^\infty (x^2-1)^{\frac{1}{2}\lambda}(x-u)^{\mu-1}Q_\nu^{-\lambda}(x)dx = \Gamma(\mu)e^{\mu\pi i}(u^2-1)^{\frac{1}{2}\lambda+\frac{1}{2}\mu}Q_\nu^{-\lambda-\mu}(u), \tag{8.2}$$

$$|\arg(u-1)| < \pi, \quad 0 < \operatorname{Re}\mu < 1+\operatorname{Re}(\nu-\lambda).$$

Setting $u=\cosh\rho$ and using (8.2), we have

(8.3)

$$\int_\rho^\infty \frac{(\sinh r)^{\lambda+1}}{(\cosh r-\cosh\rho)^{1-\mu}} Q_\nu^{-\lambda}(\cosh r)dr = \Gamma(\mu)e^{\mu\pi i}(\sinh\rho)^{\lambda+\mu}Q_\nu^{-\lambda-\mu}(\cosh\rho).$$

Setting $z=\cosh\rho$ and using (3.4), we have

$$\begin{aligned} &Q_\nu^\mu(\cosh\rho)\\ &= e^{i(\pi\mu)}2^{-\nu-1}\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)}\sinh^{-\mu}\rho\int_0^\pi(\cosh\rho+\cos t)^{\mu-\nu-1}(\sin t)^{2\nu+1}dt. \end{aligned} \tag{8.4}$$

Proof of Lemma 8.4. We firstly show

$$(-25/4-\Delta_{\mathbb{H}})^{-1} = \frac{3\sqrt{2}}{2^6\pi^3}(\sinh\rho)^{-4}\int_0^\pi(\cosh\rho+\cos t)^{\frac{3}{2}}dt. \tag{8.5}$$

In fact, by (3.7),

$$(-25/4-\Delta_{\mathbb{H}})^{-1} = \frac{\sqrt{\pi}}{(2\pi)^{\frac{7}{2}}}\int_\rho^{+\infty}\frac{\sinh r}{\sqrt{\cosh r-\cosh\rho}}\left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^2\frac{1}{\sinh r}dr. \tag{8.6}$$

By (8.4), we have

$$\begin{aligned} \left(-\frac{1}{\sinh r}\frac{\partial}{\partial r}\right)^2\frac{1}{\sinh r} &= \frac{2\cosh^2 r+1}{\sinh^5 r} = \frac{2}{\pi}\cdot\frac{1}{(\sinh r)^5}\int_0^\pi(\cosh r+\cos t)^2dt\\ &= \frac{2}{\pi}\cdot e^{-\frac{7\pi}{2}i}2^{\frac{3}{2}}\frac{\Gamma(3/2)}{\Gamma(5)}(\sinh r)^{-\frac{5}{2}}Q_{-1/2}^{5/2}(\cosh r). \end{aligned} \tag{8.7}$$

Therefore, by (8.6), (8.7) and (8.3), we have

$$\begin{aligned} &(-25/4-\Delta_{\mathbb{H}})^{-1}\\ &= \frac{\sqrt{\pi}}{(2\pi)^{\frac{7}{2}}}\frac{2}{\pi}\cdot e^{-\frac{7\pi}{2}i}2^{\frac{3}{2}}\frac{\Gamma(3/2)}{\Gamma(5)}\int_\rho^{+\infty}\frac{(\sinh r)^{-\frac{3}{2}}}{\sqrt{\cosh r-\cosh\rho}}Q_{-1/2}^{5/2}(\cosh r)dr\\ &= (2\pi)^{-3}e^{-2\pi i}(\sinh\rho)^{-2}Q_{-1/2}^2(\cosh\rho). \end{aligned} \tag{8.8}$$

Combining (8.4) and (8.8) yields (8.5).

Next choosing $\nu=\frac{3}{2}$ in (3.5), we have

$$(-4-\Delta_{\mathbb{H}})^{-1} = \frac{(2\pi)^{-3}\Gamma(4)}{2^2\Gamma(2)}(\sinh\rho)^{-4}\int_0^\pi(\sin t)^3dt = \frac{1}{4\pi^3}(\sinh\rho)^{-4}.$$

Therefore,

$$
\begin{aligned}
&\big[\big(-25/4-\Delta_{\mathbb{H}}\big)\big(-4-\Delta_{\mathbb{H}}\big)\big]^{-1}-\frac{1}{\gamma_6(4)}\cdot\frac{1}{(2\sinh\frac{\rho}{2})^2}\\
&=\frac{4}{9}\big[\big(-25/4-\Delta_{\mathbb{H}}\big)^{-1}-\big(-4-\Delta_{\mathbb{H}}\big)^{-1}\big]-\frac{1}{16\pi^3}\cdot\frac{1}{(2\sinh\frac{\rho}{2})^2}\\
&=\frac{1}{4\pi^3}(\sinh\rho)^{-4}\left\{\frac{4}{9}\left[\frac{3\sqrt{2}}{16}\int_0^\pi(\cosh\rho+\cos t)^{\frac{3}{2}}dt-1\right]-\left(\sinh\frac{\rho}{2}\right)^2\left(\cosh\frac{\rho}{2}\right)^4\right\}\\
&=\frac{1}{4\pi^3}(\sinh\rho)^{-4}\left\{\frac{4}{9}\left[\frac{3\sqrt{2}}{16}\int_0^\pi\left(1+2\sinh^2\frac{\rho}{2}+\cos t\right)^{\frac{3}{2}}dt-1\right]\right.\\
&\qquad\qquad\left.-\left(\sinh\frac{\rho}{2}\right)^2\left(1+2\sinh^2\frac{\rho}{2}\right)^2\right\}\\
&=\frac{1}{4\pi^3}(\sinh\rho)^{-4}\psi\left(\sinh^2\frac{\rho}{2}\right),
\end{aligned}
$$

where

$$
\psi(r)=\frac{4}{9}\left[\frac{3\sqrt{2}}{16}\int_0^\pi(1+2r+\cos t)^{\frac{3}{2}}dt-1\right]-r(1+r)^2,\quad r\geq 0.
$$

We compute

$$
\begin{aligned}
\psi'(r)&=\frac{\sqrt{2}}{4}\int_0^\pi(1+2r+\cos t)^{\frac{1}{2}}dt-(1+r)(1+3r);\\
\psi''(r)&=\frac{\sqrt{2}}{4}\int_0^\pi(1+2r+\cos t)^{-\frac{1}{2}}dt-4-6r.
\end{aligned}
$$

It is easy to check that

$$
\psi(0)=\psi'(0)=0,\quad \lim_{r\to 0+}\psi''(r)=+\infty.
$$

Therefore, for small enough $r>0$, we have $\psi(r)>0$. Thus

$$
\begin{aligned}
&\big[\big(-25/4-\Delta_{\mathbb{H}}\big)\big(-4-\Delta_{\mathbb{H}}\big)\big]^{-1}-\frac{1}{\gamma_6(4)}\cdot\frac{1}{(2\sinh\frac{\rho}{2})^2}\\
&\quad=\frac{1}{4\pi^3}(\sinh\rho)^{-4}\psi\left(\sinh^2\frac{\rho}{2}\right)>0,\quad 0<\rho<\epsilon,
\end{aligned}
$$

where $\epsilon>0$ is small enough. □

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