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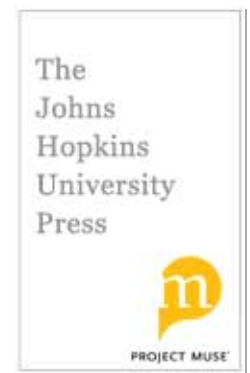
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# INFINITE BOUNDARY VALUE PROBLEMS FOR CONSTANT MEAN CURVATURE GRAPHS IN $\mathbb{H}^2 \times \mathbb{R}$ AND $\mathbb{S}^2 \times \mathbb{R}$

By LAURENT HAUSWIRTH, HAROLD ROSENBERG, and JOEL SPRUCK

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*Abstract.* We extend the classical existence and uniqueness theory of Jenkins-Serrin ( $H = 0$ ) and Spruck ( $H > 0$ ) for the constant mean curvature equation over a domain in  $\mathbb{R}^2$ , to domains in  $\mathbb{H}^2$  or  $\mathbb{S}^2$ . This theory allows prescribed boundary data including plus or minus infinity on entire arcs of the boundary. Necessarily these arcs must have curvature  $+2H$  or  $-2H$  with respect to the domain. We give necessary and sufficient conditions for existence in terms of so called admissible polygons. The key idea, as in previous proofs, is to study the “flux” of monotone increasing and decreasing sequences of solutions.

**1. Introduction.** In this paper we study constant mean curvature graphs in  $\mathbb{M} \times \mathbb{R}$  where  $\mathbb{M} = \mathbb{H}^2$  or  $\mathbb{S}^2$  the hyperbolic plane of curvature  $-1$  or the 2-sphere of curvature  $1$ . Let  $D$  be a bounded domain in  $\mathbb{M}$  with piecewise smooth boundary. We prescribe continuous data on the smooth arcs of boundary  $D$ , which may be  $+\infty$  on some arcs and  $-\infty$  on others. Our purpose is to solve the Dirichlet problem for the constant mean curvature equation in  $D$ , assuming the prescribed boundary data. More precisely, we seek a smooth function  $u$  defined in  $D$ , whose graph  $S = \{(x, u(x)) : x \in D\}$  in  $\mathbb{M} \times \mathbb{R}$  (with the standard product metric) has mean curvature  $H$  and which takes on the prescribed data on  $\partial D$ . The domains  $D$  we consider and the geometric properties that guarantee the existence of a solution to this Dirichlet problem, are given in section 7.

When  $\mathbb{M} = \mathbb{R}^2$  and  $H = 0$ , this is the classical theory of Jenkins-Serrin [3]. This theory was extended to  $H \neq 0$  by Spruck [9], again for  $\mathbb{M} = \mathbb{R}^2$ . For  $H = 0$ , the Jenkins-Serrin theory was extended to  $\mathbb{M} = \mathbb{H}^2$  by Nelli and Rosenberg [4] and to arbitrary Riemannian surfaces  $\mathbb{M}$  by Pinheiro [6]. Our methods also work in this case.

We shall always assume that  $H > 0$  with respect to the upward unit normal of  $S$ . Then if  $u$  tends to  $+\infty$  for any approach to a boundary arc  $\gamma$ , then necessarily its curvature  $\kappa(\gamma) = 2H$  is constant, while if  $u$  tends to  $-\infty$  on  $\gamma$  then  $\kappa(\gamma) = -2H$  (see Theorem 4.8). Thus we must deal with nonconvex domains  $D$  with  $\partial D$  piecewise  $C^2$  and consisting of three sets of open arcs  $\{A_i\}$ ,  $\{B_i\}$  and  $\{C_i\}$  satisfying  $\kappa(A_i) = 2H$ ,  $\kappa(B_i) = -2H$  and  $\kappa(C_i) \geq 2H$  respectively. The generalized Dirichlet problem is to find a solution of (2.1) in  $D$  taking on the

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boundary values  $+\infty$  on the  $A_i$ ,  $-\infty$  on the  $B_i$  and arbitrary continuous boundary data on the  $C_i$ . A precise definition of the admissible domains  $D$  is given in Definition 7.1 of section 7. When the family  $\{C_i\}$  is nonempty, we give in Theorem 7.11 necessary and sufficient conditions that there exist a unique solution. These are the generalized Jenkins-Serrin flux conditions (as in [9]) formulated in terms of admissible polygons (see Definition 7.4 of section 7). When the family  $\{C_i\}$  is empty we give in Theorem 7.12 necessary and sufficient conditions for ‘‘Scherk type’’ solutions to exist; these are unique up to a vertical translation (see Theorem 7.13). Examples of this type are constructed for  $\mathbb{H}^2$  and  $H \leq \frac{1}{2}$  in section 8.

The proof of these results is long and involved and essentially follows that of [9] using the existence theory and interior gradient estimates of [10] as basic tools. As in all previous proofs, the key idea is to study the flux of monotone increasing and decreasing sequences of solutions along arcs where they diverge. This is carried out in sections 5 and 6. Section 2 contains an important general maximum principle which in conjunction with the special barriers of section 4 allows us to analyse the flux. Section 3 formulates a general Perron existence theorem which is a basic tool for the existence theory of section 7. In some sense, the Scherk type examples with  $H = \frac{1}{2}$  in  $\mathbb{H}^2 \times \mathbb{R}$  are the most interesting as they can be constructed over arbitrarily large domains. We hope in future work to use these to construct interesting complete graphs.

**2. The mean curvature equation and a general maximum principle.** Let  $ds^2 = \sum_{i,j=1}^2 \sigma_{ij} dx_i dx_j$  is a local Riemannian metric on  $\mathbb{M}$  and denote by  $(\sigma^{ij})$  the inverse matrix of  $(\sigma_{ij})$ . Then  $\mathbb{M} \times \mathbb{R}$  is given with the product metric  $ds^2 + dt^2$  where  $t$  is a coordinate for  $\mathbb{R}$  and (see [10]) if  $S = \{\text{graph } u\}$  has constant mean curvature  $H$ , the height function  $u(x) \in C^2(\Omega)$  satisfies the divergence form equation

$$(2.1) \quad Mu := \operatorname{div} \frac{\nabla u}{W} = 2H, \quad W = (1 + |\nabla u|^2)^{1/2}$$

where the divergence and gradient  $\nabla u$  are taken with respect to the metric on  $\mathbb{M}$ . Equivalently, equation (2.1) can be written in nondivergence form

$$(2.2) \quad Mu = \frac{1}{W} \sum_{i,j=1}^2 g^{ij} D_i D_j u = 2H,$$

where  $D$  denotes covariant differentiation on  $\mathbb{M}$  and

$$g^{ij} = \sigma^{ij} - \frac{u^i u^j}{W^2}, \quad u^i = \sum_{j=1}^2 \sigma^{ij} D_j u.$$

The main result of this section is a general maximum principle for sub and super solutions of the mean curvature operator for boundary data with a finite number of discontinuities. We follow the argument of [9] which holds in the general setting with little change. For the convenience of the reader we sketch the argument which uses the divergence structure.

LEMMA 2.1. *Let  $u^1$  and  $u^2$  be functions in  $C^2(\Omega)$ ,  $\Omega \subset \mathbb{M}$  and set  $W_k = W(\nabla u^k)$ ,  $k = 1, 2$  where  $W(p) = \sqrt{1 + p^2}$ . Then*

$$(2.3) \quad \left\langle \nabla u^1 - \nabla u^2, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} \right\rangle \geq 0$$

with equality at a point if and only if  $\nabla u^1 = \nabla u^2$ .

*Proof.* The downward unit normal  $n_i$  to  $S_i = \{\text{graph } u^i\}$  is given by

$$n_i = \frac{\nabla u^i}{W_i} - \frac{1}{W_i} e, \quad e = \frac{\partial}{\partial t}.$$

Hence using  $\langle W_i n_i, e \rangle = -1$ ,

$$\begin{aligned} \left\langle \nabla u^1 - \nabla u^2, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} \right\rangle &= \left\langle W_1 n_1 - W_2 n_2, n_1 - n_2 + \left( \frac{1}{W_1} - \frac{1}{W_2} \right) e \right\rangle \\ &= \langle W_1 n_1 - W_2 n_2, n_1 - n_2 \rangle \\ &= (W_1 + W_2)(1 - \langle n_1, n_2 \rangle) \\ &= \frac{W_1 + W_2}{2} \|n_1 - n_2\|^2 \geq 0. \end{aligned}$$

(Here, the inner products are in the metric of  $\mathbb{M} \times \mathbb{R}$ .) □

THEOREM 2.2. (General maximum principle) *Let  $u^1$  and  $u^2$  satisfy  $Mu^1 \geq nH \geq Mu^2$  in a bounded domain  $D \subset \mathbb{M}$ . Suppose that  $\liminf (u^2 - u^1) \geq 0$  for any approach to  $\partial D$  with the possible exception of a finite number of points of  $\partial D$ . Then  $u^2 \geq u^1$  with strict inequality unless  $u^2 \equiv u^1$ .*

*Proof.* Let  $N$  and  $\varepsilon$  be positive constants with  $N$  large and  $\varepsilon$  small. Define

$$\varphi = \begin{cases} N - \varepsilon & \text{if } u^1 - u^2 \geq N \\ u^1 - u^2 - \varepsilon & \text{if } \varepsilon < u^1 - u^2 < N \\ 0 & \text{if } u^1 - u^2 \leq \varepsilon. \end{cases}$$

Then  $\varphi$  is Lipschitz and  $0 \leq \varphi < N$  and  $\nabla \varphi = \nabla u^1 - \nabla u^2$  in the set where  $\varepsilon < u^1 - u^2 < N$  and  $\nabla \varphi = 0$  almost everywhere in the complement of this set. About each exceptional boundary point  $P_i$ ,  $i = 1, \dots, k$  we construct a ball

of radius  $\varepsilon$  and obtain a domain  $D_\varepsilon \subset D$  bounded by spherical boundaries  $C_i$ ,  $i = 1, \dots, k$  and a subset of  $\partial D$  which we simply denote by  $\Gamma$ . Evidently,  $\varphi \equiv 0$  in a neighborhood of  $\Gamma$ . Now let

$$J = \int_{\sum C_i} \varphi \left\langle \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2}, \nu \right\rangle ds$$

where  $\nu$  is the exterior normal to  $\partial D_\varepsilon$ . Then

$$(2.4) \quad J \leq 4\pi\varepsilon kN.$$

Since

$$\operatorname{div} \left( \varphi \left( \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} \right) \right) = \left\langle \nabla \varphi, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} \right\rangle + \varphi \left( \operatorname{div} \frac{\nabla u^1}{W_1} - \operatorname{div} \frac{\nabla u^2}{W_2} \right)$$

almost everywhere in  $D$ , we can apply the divergence theorem and obtain

$$(2.5) \quad J = \int_{D_\varepsilon} \left( \left\langle \nabla \varphi, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} \right\rangle + \varphi \left( \operatorname{div} \frac{\nabla u^1}{W_1} - \operatorname{div} \frac{\nabla u^2}{W_2} \right) \right) dV.$$

By our assumptions, the last term of the integrand in (2.5) is nonnegative so by (2.3), (2.4), (2.5) we have

$$(2.6) \quad 0 \leq \int_{D_\varepsilon} \left\langle \nabla \varphi, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} \right\rangle dV \leq 4\pi\varepsilon kN.$$

As  $\varepsilon$  decreases to zero, the sets  $D_\varepsilon$  are expanding and thus  $\nabla u^1 \equiv \nabla u^2$  in the set  $\{0 < u^1 - u^2 < N\}$ . Since  $N$  is arbitrary,  $\nabla u^1 \equiv \nabla u^2$  whenever  $u^1 > u^2$ . It follows that  $u^1 \equiv u^2 + \text{positive constant}$  in any nontrivial component of the set  $\{u^1 > u^2\}$ . Suppose there exists one such component. Then the maximum principle ensures  $u^1 \equiv u^2 + \text{positive constant}$  in  $D$  and by the assumptions of the theorem, the constant must be nonpositive, a contradiction.

*Remark 2.3.* With a little more effort using a covering argument as in [5], we can allow the exceptional set  $E$  of points on  $\partial D$  to have one dimensional Hausdorff measure  $H_1(E) = 0$ .

**3. Existence and compactness theorems.** In this section we state the basic existence and compactness theorems we shall use in our study of the Dirichlet problem with infinite boundary data.

**THEOREM 3.1.** (Compactness theorem) *Let  $\{u_n\}$  be a uniformly bounded sequence of solutions of (2.1) in a bounded domain  $D$ . Then there exists a subsequence*

which converges uniformly on compact subsets (in any topology) to a solution of (2.1) in  $D$ .

Theorem 3.1 follows from the interior gradient estimate of [10] (see (6.22) at the beginning of section 6 for a precise statement) and a standard argument (see Corollary 16.7 and Theorem 16.8 of [2]) using Schauder theory and the Arzela-Ascoli theorem.

We next turn to existence theorems. For simplicity, we consider only bounded piecewise  $C^2$  domains  $D$ . For  $P \in \partial D$  define the outer curvature  $\hat{\kappa}(P)$  to be the supremum of all (inward) normal curvatures of  $C^2$  curves passing through  $P$  and locally supporting  $D$ . If no such curve exists we define  $\hat{\kappa}(P)$  to be  $-\infty$ . Note that  $\hat{\kappa}(P) = \kappa(P)$  at all regular points of  $\partial D$ .

**THEOREM 3.2.** (Existence theorem) *Let  $D$  be a domain piecewise  $C^1$ . Suppose that  $\hat{\kappa}(P) \geq 2H$  for all  $P \in \partial D$  except for a finite set  $E$  of exceptional corner points of  $\partial D$ .*

(a) *If  $E = \{\emptyset\}$  there is a unique solution of (2.1) in  $D$  taking on arbitrarily assigned continuous boundary data on  $\partial D$ .*

(b) *If  $E \neq \{\emptyset\}$  there is a unique solution of (2.1) in  $D$  taking on arbitrarily assigned continuous boundary data on  $\partial D \setminus E$  if either*

- (i)  $\mathbb{M} = \mathbb{H}^2$  and  $H \leq \frac{1}{2}$ , or
- (ii) *there is a bounded subsolution of (2.1) in  $D$ .*

*Remark 3.3.* Theorem 3.2 is unusual in that existence and uniqueness holds for certain special nonconvex domains. Note that  $D$  may even be multiply connected. A simple example is provided by an annulus with outer boundary a circle and inner boundary a quadrilateral composed of four (or an even number) of circular arcs (convex toward the annulus). If all the circles have curvature at least  $2H$ , Theorem 3.2 guarantees a unique solution for arbitrary continuous boundary data. Note that the solution is not required to be continuous at the corner points and in general cannot be so. By a simple approximation argument we can also allow a finite number of discontinuities in the boundary data on  $\partial D \setminus E$ , that is on the part of the boundary satisfying the generalized curvature condition.

*Example 3.4.* (Lens domain) A simple but useful example is provided by a lens domain  $L$  bounded by an arc  $\gamma$  of curvature  $2H$  and its geodesic reflection  $\gamma^*$ . In  $\mathbb{S}^2$  for all  $H > 0$  and in  $\mathbb{H}^2$  for  $H > \frac{1}{2}$  we must restrict the length of  $\gamma$  to half that of the geodesic circle of curvature  $2H$ . However in  $\mathbb{H}^2$  for  $0 < H \leq \frac{1}{2}$  there are no restrictions needed. In all cases, a subsolution of (2.1) exists, so there is existence and uniqueness for the Dirichlet problem in the lens domain.

**LEMMA 3.5.** *Theorem 3.2 holds when  $\partial D \in C^{2+\alpha}$  and  $\kappa(P) \geq 2H$  for all  $P \in \partial D$ .*

*Proof.* The existence theory developed in [10] (specifically Theorems 1.4 and 5.4) specialized to dimension 2 and  $\mathbb{M} = \mathbb{S}^2$  or  $\mathbb{M} = \mathbb{H}^2$  (with  $H \geq \frac{1}{2}$ )

gives the existence of a unique solution  $u \in C^{2+\alpha}(\overline{D})$  of (2.1) under the classical solvability condition  $\kappa \geq 2H$ . For  $\mathbb{M} = \mathbb{H}^2$  and  $H < \frac{1}{2}$  we need in addition a special local barrier construction (see Corollary 4.3 of the next section) and then the argument of [10] works as before.

Using Lemma 3.5 we construct local barriers needed in the proof of Theorem 3.2.

**COROLLARY 3.6.** *Let  $\gamma$  be an arc of constant curvature  $2H$  and  $Q$  an interior point of  $\gamma$ . Let  $\mathcal{N}_\delta \subset B_\delta(Q)$  be a smooth (convex) domain obtained by smoothing the convex domain bounded by  $\gamma$  and  $\partial B_\delta(Q)$ . Consider smooth boundary data  $f \leq 0$  on  $\partial \mathcal{N}_\delta$  with  $f \equiv 0$  in a neighborhood of  $Q$  and  $f \equiv -M$  on a neighborhood of  $\partial B_\delta(Q)$ . Then there is a smooth solution  $w^-$  of (2.1) in  $\overline{\mathcal{N}_\delta}$  with boundary values  $f$ .*

The family  $w^-$  and  $w^+ = -w^-$  depending on  $M, \delta$  are local lower and upper barriers at  $Q$ .

*Proof of Theorem 3.2.* For the existence in case (a), we approximate  $D$  by smooth (convex) domains  $D_n \subset D$  satisfying  $\kappa(\partial D_n) \geq 2H$  by rounding each corner point of  $\partial D$ . Extend the boundary data  $f$  continuously to all of  $D$  and let  $f_n$  be smooth functions in  $\overline{D_n}$  converging uniformly to  $f$ . Then Theorems 1.4 and 5.4 of [10] give a unique smooth solution  $u_n$  in  $D_n$  with  $u_n = f_n$  on  $\partial D_n$  and each  $u_n$  is uniformly bounded independent of  $n$  (see also [1]). Thus by the compactness theorem, a subsequence of the  $u_n$  converges uniformly on compact subsets to a solution  $u$  of (2.1) in  $D$ . It remains to show that  $u = f$  on  $\partial D$ . Fix  $P \in \partial D$ ,  $Q \in \partial D_n$  with  $d(Q, P) < \delta$ . Given  $\varepsilon > 0$  choose  $\delta > 0$  such that  $|f_n(x) - f_n(Q)| < \varepsilon$  and  $|f_n(Q) - f(P)| < \varepsilon$  if  $d(x, Q) < \delta$  and  $n$  large. Now let  $\gamma$  be an arc of constant curvature  $2H$  that supports  $\partial D_n$  at  $Q$  and let  $w^+ = -w^- = w(x)$  be the upper and lower barriers in  $\mathcal{N}_\delta$  given by Corollary 3.6 with  $M = 2 \sup |u_n|$ . Then by the maximum principle

$$-w(x) - 3\varepsilon \leq u_n(x) - f(P) \leq w(x) + 3\varepsilon \text{ in } \mathcal{N}_\delta \cap D_n,$$

Hence there is a uniform modulus of continuity  $\sigma(t)$  so that  $|u_n(x) - f(P)| < \sigma(d(x, P))$  for any  $P \in \partial D$  and  $x \in D_n$ . It follows that  $u \in C(\overline{D})$  and  $u = f$  on  $\partial D$ .

For part (b), we use Perron's method as in [8]. We note that the existence of a bounded subsolution in case  $\mathbb{M} = \mathbb{H}^2$  and  $H \leq \frac{1}{2}$  is automatic since there are global radial solutions. Otherwise, we assume such a bounded subsolution exists. Note that the constants are always supersolutions. This being the case there is a Perron solution  $u$  in  $D$  which achieves the assigned boundary data on  $\partial D \setminus E$  because of the existence of local barriers.

In all cases, the solution is unique by the General maximum principle, Theorem 2.2.

**4. Special barriers and the asymptotic behavior of solutions with infinite boundary values.** We begin this section with the construction of special barriers in case  $\mathbb{M} = \mathbb{H}^2$  and  $H \leq \frac{1}{2}$ . For the half-space model of  $\mathbb{H}^2 = \{(x, y): y \geq 0\}$  with metric  $ds^2 = \frac{dx^2 + dy^2}{y^2}$ , note that

$$(4.7) \quad Mu = y^2 \left\{ \frac{\partial}{\partial x} \left( \frac{u_x}{\sqrt{1 + y^2(u_x^2 + u_y^2)}} \right) + \frac{\partial}{\partial y} \left( \frac{u_y}{\sqrt{1 + y^2(u_x^2 + u_y^2)}} \right) \right\}.$$

LEMMA 4.1. For  $u = f(\phi(x) - y)$ ,

$$(4.8) \quad Mu = \frac{y^2}{(1 + y^2 f'^2 (1 + \phi'^2))^{\frac{3}{2}}} \left\{ f' \phi'' + f'' (1 + \phi'^2) + y^2 f'^3 \phi'' + y f'^3 (1 + \phi'^2) \right\}.$$

In particular for  $\phi(x) = 2H + \sqrt{1 - x^2}$  and  $f(t) = -\frac{1}{a} \log |t|$  for  $0 < |t| < \frac{\varepsilon_0}{a}$  ( $\varepsilon_0$  sufficiently small independent of  $a > 0$ )

$$(4.9) \quad Mu = 2H \left\{ 1 + \frac{a^2 t}{2Hy} \left( \sqrt{1 - x^2} - \frac{y}{a^2} \right) + O((at)^2) \right\},$$

where  $t = 2H + \sqrt{1 - x^2} - y$ . Hence  $u$  tends to  $+\infty$  on the curve  $C = \{(x, y): y = \phi(x)\}$  and in a neighborhood of  $(0, 1 + 2H)$ , if  $a > \sqrt{1 + 2H}$ ,  $u$  is a subsolution for  $t > 0$  and a supersolution for  $t < 0$  while if  $a < \sqrt{1 + 2H}$ ,  $u$  is a supersolution for  $t > 0$  and a subsolution for  $t < 0$ .

*Proof.* Equation (4.8) is a straightforward computation. To derive (4.9) we make a Taylor expansion about  $t = 0$  (using  $f' = -\frac{1}{at}$ ,  $f'' = af'^2$ ,  $\phi'' = -(1 + \phi'^2)^{\frac{3}{2}} = -(1 - x^2)^{-\frac{3}{2}}$ ):

$$\begin{aligned} Mu &= \left( 1 - \frac{3(1 - x^2)}{2y^2} a^2 t^2 + O(a^4 t^4) \right) \left( 2H - t + \frac{\sqrt{1 - x^2}}{y} a^2 t + \frac{a^2 t^2}{y} \right) \\ &= 2H + \left( a^2 \frac{\sqrt{1 - x^2}}{y} - 1 \right) t + O(a^2 t^2). \end{aligned}$$

*Remark 4.2.* For  $2H < 1$  the curve  $C = \{(x, y): y = \phi(x)\}$  is a so called equidistant circle and has constant curvature  $2H$  (For  $2H = 1$  the curve  $C$  is half a horocycle). Note that the translates  $C_t = \{(x, y): y = \phi(x) - t\}$ ,  $t > 0$  are also equidistant circles of smaller curvature  $2H - t$ .

COROLLARY 4.3. Let  $D$  be a domain in  $\mathbb{H}^2$  bounded in part by an arc  $\Gamma$  with  $\hat{\kappa} \geq 2H$  (for  $H \leq \frac{1}{2}$ ). Given  $M > 0$  large, let  $C_{t_0}$  be a supporting equidistant circle  $C$  at  $P \in \Gamma$  of constant curvature  $2H - t_0$  ( $0 < t_0 \leq \frac{\varepsilon_0}{2a} e^{-aM}$ ). For  $C'$  a translate of  $C$  of curvature  $2H - e^{aM} t_0$ , let  $N(t_0, M)$  be the subdomain of  $\bar{D}$  bounded by  $\Gamma$



and  $C'$ . Then there is a subsolution  $w < 0$  of (2.1) in  $N$  with  $w(P) = 0$  and  $w = -M$  on  $C'$ .

*Proof.* Let  $P = (0, 2H)$  and  $\{y = \phi(x)\}$  be the equidistant circle of Remark 4.2 so that  $C_{t_0}$  is the translate  $\{y = \phi(x) - t_0\}$ . Then  $w = -\frac{1}{a} \log \frac{\phi(x)-y}{t_0}$  has the required properties.  $\square$

We next construct barriers (with infinite slope) using the (oriented) distance function  $d(x)$  to an arc  $\Gamma$  in  $M$ . Let  $w = h(d)$ ; then as in [10] (with  $n = 2$ ) from the nondivergence form (2.2) of the equation

$$\begin{aligned}
 (4.10) \quad Mw &= \frac{1}{W} \left( \sigma^{ij} - \frac{w^i w^j}{W^2} \right) (h' D_i D_j d + h'' D_i d D_j d) \\
 &= \frac{1}{\sqrt{1+h^2}} \left( h' \Delta d + \frac{h''}{(1+h^2)} \right) \\
 &= \frac{h''}{(1+h^2)^{\frac{3}{2}}} - \frac{h'}{\sqrt{1+h^2}} \kappa(x),
 \end{aligned}$$

where  $\kappa(x)$  is the inward curvature of the level set of  $d(x)$  passing through  $x$ .

LEMMA 4.4. (i) Let  $\Gamma$  be a  $C^2$  arc with  $\kappa(\Gamma) \leq 2\alpha < 2H$  with respect to the “interior” unit normal  $-n$ . Then in a sufficiently small neighborhood  $\mathcal{N}$  of any interior point of  $\Gamma$ , there is a supersolution  $w$  of (2.1) with exterior normal derivative  $\frac{\partial w}{\partial n} = +\infty$  on  $\Gamma$ .

(ii) If  $\kappa(\Gamma) \leq -2\alpha < -2H$  with respect to the “interior” unit normal  $-n$ , then there is a subsolution  $w$  of (2.1) with exterior normal derivative  $\frac{\partial w}{\partial n} = -\infty$  on  $\Gamma$ .

*Proof.* Let  $h(t) = -\sqrt{\frac{2t}{\varepsilon}}$ . Then  $\frac{h''}{(1+h^2)^{\frac{3}{2}}} = \varepsilon(1+2\varepsilon t)^{-\frac{3}{2}}$ ,  $\frac{h'}{\sqrt{1+h^2}} = -(1+2\varepsilon t)^{-\frac{1}{2}}$ , so from (4.10)  $w = h(d)$  satisfies

$$Mw = \varepsilon(1+2\varepsilon d)^{-\frac{3}{2}} + (1+2\varepsilon d)^{-\frac{1}{2}} \kappa(x) \leq (1+2\varepsilon d)^{-\frac{1}{2}} (2\alpha + 2\varepsilon) < 2H$$

for  $\varepsilon$  and  $d$  small enough, proving (i). For part (ii) let  $w = -h(d)$ . Then

$$Mw = -\varepsilon(1+2\varepsilon d)^{-\frac{3}{2}} - (1+2\varepsilon d)^{-\frac{1}{2}} \kappa(x) \geq (1+2\varepsilon d)^{-\frac{1}{2}} (2\alpha - 2\varepsilon) > 2H$$

for  $\varepsilon$  and  $d$  small enough, proving (ii).  $\square$

Remark 4.5. For  $\varepsilon = \frac{1}{2}|\alpha - H|$  the barriers of the lemma are defined in an annulus  $\{0 < d(x) < \frac{\varepsilon}{C}\}$  of size proportional to  $\varepsilon$ . For  $M = \mathbb{R}^2, \mathbb{S}^2$  and  $\mathbb{H}^2$  (here we must restrict  $H > \frac{1}{2}$ ), Delaunay’s onduloids provide explicit such barriers which are radial. For later use we note also that if  $\Gamma$  has constant curvature  $-2\alpha = -2(H + \varepsilon)$ , then  $2H \leq Mw \leq 2H + \varepsilon$  so  $w$  is an approximate solution in an annulus of size proportional to  $\varepsilon$ .

These special barriers are important because of the following well-known variant of the maximum principle due to Finn.

LEMMA 4.6. *Let  $D$  be a domain whose boundary is the union of two closed sets  $\gamma_1$  and  $\gamma_2$  with  $\gamma_2 \in C^1$ . Let  $u_1 \in C^2(D) \cap C^1(D \cup \gamma_2)$  and  $u_2 \in C^2(D) \cap C^0(\bar{D})$  satisfy  $Mu_1 \geq Mu_2$  in  $D$  and suppose  $\frac{\partial u_2}{\partial \nu} = +\infty$  ( $\nu$  is the outer normal) at every point of  $\gamma_2$ . Then if  $\liminf (u_2 - u_1) \geq 0$  for any approach to  $\gamma_1$ , then  $u_2 \geq u_1$  in  $D$ .*

COROLLARY 4.7. *Let  $u$  be a solution of (2.1) in a domain  $D$  and let  $P$  be a regular point of  $\partial D$ . If  $\kappa(P) < 2H$  then  $u$  cannot tend to  $+\infty$  at  $P$ . If  $\kappa(P) < -2H$  then  $u$  cannot tend to  $-\infty$  at  $P$ .*

*Proof.* Suppose  $\kappa(P) < 2H$ . Then there is an arc  $C_0$  of curvature  $\kappa < 2H$  internally tangent to  $\partial D$  at  $P$ . Let  $C_t$ ,  $0 < t < \delta$  be inward parallel arcs to  $C_0$ , with  $\delta$  so small that all have curvature  $\kappa < 2H$ . We can then choose a domain  $N_t$  (contained in  $D$ ) bounded by  $C_t$ ,  $C_\delta$  and two geodesics joining them that are strictly interior to  $D$  (independent of  $t$ ). We can also arrange that  $N_t$  is sufficiently small that the supersolution  $w$  of Lemma 4.4 is well-defined for  $\Gamma = C_t$ . Set  $M = \sup_{\partial N_t \setminus C_t} u + \sup_{N_t} w$ . Then  $u \leq w + M$  in  $N_t$  by Lemma 4.6. Letting  $t \rightarrow 0$  implies  $u(P) < \infty$ . An analogous argument in case  $\kappa(P) < -2H$  shows  $u(P) > -\infty$ .  $\square$

We can now characterize those arcs on which solutions can tend to  $\pm\infty$ .

THEOREM 4.8. *Let  $D$  be a domain bounded in part by a  $C^2$  arc  $\gamma$  and let  $u$  be a solution of (2.1). If  $u$  tends to  $+\infty$  for any approach to interior points of  $\gamma$ , then  $\kappa(\gamma) = 2H$ , while if  $u$  tends to  $-\infty$  for any approach to interior points of  $\gamma$ , then  $\kappa(\gamma) = -2H$ .*

*Proof.* Suppose  $u \rightarrow +\infty$  on  $\gamma$ . Then by Corollary 4.7  $\kappa(\gamma) \geq 2H$ . Now suppose  $\kappa > 2H$  on some subarc  $\gamma' \subset \gamma$ . Then by shrinking  $\gamma'$  if necessary, we can find a subdomain  $\Delta \subset D$  bounded by  $\gamma'$  and an arc  $\Gamma$  with curvature (with respect to the interior of  $\Delta$ ) satisfying  $\kappa(\Gamma) < -2H$ . The maximum principle Lemma 4.6 and part (ii) of Lemma 4.4 implies  $u > w + M$  in  $\Delta$  for any constant  $M$ , a contradiction. Hence  $\kappa(\gamma) \equiv 2H$ . Now suppose  $u \rightarrow -\infty$  for any approach to interior points of  $\gamma$ . Then by Corollary 4.7  $\kappa(\gamma) \geq -2H$ . If  $\kappa > -2H$  on some subarc  $\gamma' \subset \gamma$ . Then by shrinking  $\gamma'$  if necessary, we can find a subdomain  $\Delta \subset D$  bounded by  $\gamma'$  and an arc  $\Gamma$  with curvature (with respect to the interior of  $\Delta$ ) satisfying  $\kappa(\Gamma) < 2H$ . Then Lemma 4.6 and part i) of Lemma 4.4 implies  $u < w - M$  in  $\Delta$  for any constant  $M$ , a contradiction. Thus  $\kappa(\gamma) \equiv -2H$ .  $\square$

The proof of Theorem 4.8 contains the following useful result.

LEMMA 4.9. *Let  $u$  be a solution of (2.1) in a domain  $D$  bounded in part by an arc  $\gamma$  and suppose  $m \leq u \leq M$  on  $\gamma$ . Then there is a constant  $c=c(D)$  such that for*

any compact  $C^2$  subarc  $\gamma' \subset \gamma$ ,

(i) if  $\kappa \geq 2H$  on  $\gamma'$  with strict inequality except for isolated points, there is a neighborhood  $\Delta$  of  $\gamma'$  in  $\bar{D}$  such that  $u \geq m - c$  in  $\Delta$ .

(ii) if  $\kappa > -2H$  on  $\gamma'$ , there is a neighborhood  $\Delta$  of  $\gamma'$  in  $\bar{D}$  such that  $u \leq M + c$  in  $\Delta$ .

**5. Flux formula on boundary arcs.** In this section we collect some flux formulas we will need to develop our existence theory for solutions with infinite boundary values. These formulas are exactly the same as in section 4 of [9] with almost identical proof. The only essential difference is that we use the barrier in part (ii) of Lemma 4.4 instead of Delaunay solutions. Therefore we only briefly indicate the idea of the proofs which are already contained in the proof of the general maximum principle Theorem 2.2.

Let  $u \in C^2(D) \cap C^1(\bar{D})$  be a solution of (2.1) in a domain  $D$ . Then integrating (2.1) over  $D$  gives

$$(5.11) \quad 2HA(D) = \int_{\partial D} \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds,$$

where  $\mathcal{A}(D)$  is the area of  $D$  and  $\nu$  is the outer normal to  $\partial D$ . The right-hand integral is called the flux of  $u$  across  $\partial D$ . Let  $\gamma$  be a subarc of  $\partial D$  (homeomorphic to  $[0, 1]$ ). Even if  $u$  is not differentiable on  $\gamma$  we can define the flux of  $u$  across  $\gamma$  as follows.

*Definition 5.1.* Choose  $\Gamma$  to be a simple smooth curve in  $D$  so that  $\gamma \cup \Gamma$  bounds a simply connected domain  $\Delta_\Gamma$ . We then define the flux of  $u$  across  $\gamma$  to be

$$(5.12) \quad F_u(\gamma) = 2HA(\Delta_\Gamma) - \int_\Gamma \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds.$$

The last integral in (5.12) is well defined as an improper integral. To see that this definition is independent of  $\Gamma$ , let  $\Gamma'$  be another choice of curve and consider the 2-chain  $R$  with oriented boundary  $\Gamma' - \Gamma$ . Using (2.1) and the divergence theorem on  $R$  gives

$$2HA(\Delta_{\Gamma'}) - 2HA(\Delta_\Gamma) = \int_{\Gamma'} \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds - \int_\Gamma \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds.$$

Therefore the definition makes sense. If  $u \in C^1(D \cup \gamma)$ , then  $F_u(\gamma) = \int_\gamma \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds$ .

The following lemma is now obvious.

**LEMMA 5.2.** *Let  $u$  be a solution of (2.1) in a domain  $D$  and let  $\Gamma$  be a piecewise  $C^1$  curve in  $\bar{D}$ . Then*

$$2HA(D) = \int_{\partial D} \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds \text{ and } \left| \int_\Gamma \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds \right| \leq |\Gamma|.$$

LEMMA 5.3. *Let  $D$  be a domain bounded in part by a piecewise  $C^2$  arc  $\Gamma$  satisfying  $\hat{\kappa}(\Gamma) \geq 2H$ . Let  $u$  be a solution of (2.1) in  $D$  which is continuous on  $\Gamma$ . Then*

$$(5.13) \quad \left| \int_{\Gamma} \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds \right| < |\Gamma|.$$

*Proof.* It suffices to prove (5.13) for a small subarc  $\gamma$  of  $\Gamma$ . To this end let  $P \in \Gamma$  and let  $D_\varepsilon = D \cap B_\varepsilon(P)$ . Then by Theorem 3.2 there is a solution  $v$  of (2.1) in  $D_\varepsilon$  with  $v = u + 1$  on  $\gamma$  and  $v = u$  on the remainder of the boundary. Set  $w = v - u$ . Then as in the proof of the general maximum principle,

$$0 < \int_{D_\varepsilon} \left\langle \nabla w, \frac{\nabla v}{W_v} - \frac{\nabla u}{W_u} \right\rangle dV = \int_{\gamma} \left\langle \frac{\nabla v}{W_v} - \frac{\nabla u}{W_u}, \nu \right\rangle ds.$$

Hence  $F_u(\gamma) < F_v(\gamma) \leq |\gamma|$ . □

LEMMA 5.4. *Let  $D$  be a domain bounded in part by an arc  $\gamma$  and let  $u$  be a solution of (2.1) in  $D$ . Then*

- (i) *if  $u$  tends to  $+\infty$  on  $\gamma$ , we have  $\int_{\gamma} \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds = |\gamma|$ .*
- (ii) *if  $u$  tends to  $-\infty$  on  $\gamma$ , we have  $\int_{\gamma} \left\langle \frac{\nabla u}{W}, \nu \right\rangle ds = -|\gamma|$ .*

*Proof.* We sketch the proof of (i) which is a slight modification of Lemma 4.3 of [9]; part (ii) is similar. By Theorem 4.8,  $\kappa(\gamma) = 2H$ . Fix  $\varepsilon > 0$  and let  $\delta$  be arbitrarily small. Let  $\gamma'$  be a subarc of  $\gamma$  of length a small multiple of  $\sqrt{\varepsilon}$ . Choose an arc  $\Lambda$  of constant curvature  $-2\alpha = -2(H + \varepsilon)$  (with respect to its normal pointing exterior of  $D$ ) so that the distance between  $\Lambda$  and  $\gamma'$  satisfies  $\delta < d(\Lambda, \gamma') \leq \delta + C|\gamma'|^2$ . Then by Remark 4.5, the subsolution  $w$  of Lemma 4.4 part (ii) (with  $\Gamma$  replaced by  $\Lambda$ ) is well defined in an annulus of size proportional to  $\varepsilon$  (with  $\gamma'$  strictly interior) and  $w$  is an approximate solution with flux

$$(5.14) \quad F_w(\gamma') \geq (1 - \varepsilon(\delta + C|\gamma'|^2))|\gamma'|.$$

Now choose a simple arc  $\Gamma$  joining the endpoints of  $\gamma'$  so that  $\gamma' \cup \Gamma$  bounds a simply connected domain  $\Delta$  which is contained in the annulus. Set  $\varphi = u_1 - u_2$  where  $u_1 = u$  and  $u_2 = w$ . We may assume  $\varphi > 0$ . Let  $M_0$  be chosen so large that the linear measure of  $\Gamma \cap \{\varphi > M_0\}$  is less than  $\frac{\varepsilon}{8}$ . Let  $\Gamma_0$  be the subarc of  $\Gamma$  with endpoints at arc length distance  $\frac{\varepsilon}{16}$  from the endpoints of  $\Gamma$ . For  $M > \sup_{\Gamma_0} \varphi$ , let  $\Delta_M$  be the component of the set  $\Delta \cap \{\varphi < M\}$  containing  $\Gamma_0$  in its closure. Then  $\Delta_M$  will be bounded by a curve  $\gamma_M$  with endpoints on  $\Gamma$  close to those of  $\gamma'$  (on which  $\varphi = M$ ), by subarcs  $\Gamma_M \subset \Gamma$  and possibly by other curves  $\tilde{\gamma}_M$  (with endpoints on  $\Gamma$ ) on which  $\varphi = M$ . Further we may always choose  $M$  so that  $\partial\Delta_M$  is piecewise smooth.

Now let

$$J = \int_{\Delta_M} \left\langle \nabla \varphi, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right\rangle dV.$$

Since

$$\operatorname{div} \varphi \left( \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right) = \left\langle \nabla \varphi, \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2} \right\rangle + \varphi \left( \operatorname{div} \frac{\nabla u_1}{W_1} - \operatorname{div} \frac{\nabla u_2}{W_2} \right)$$

in  $D$ , we can apply the divergence theorem and obtain

$$\begin{aligned} (5.15) \quad J &= M \int_{\gamma_M} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle ds + \int_{\Gamma_M} \varphi \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle ds \\ &\quad + M \int_{\tilde{\gamma}_M} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle ds + O(M\varepsilon|\Delta|) \\ &= J_1 + J_2 + J_3 + O(M\varepsilon|\Delta|). \end{aligned}$$

We can estimate  $J_2$  as follows:

$$(5.16) \quad J_2 \leq \frac{\varepsilon}{4} M + 2|\Gamma|M_0.$$

To estimate  $J_3$  we complete  $\tilde{\gamma}_M$  to closed curves by subarcs  $\tilde{\Gamma}_M$  of  $\Gamma_M$ ; then

$$J_3 = M \int_{\tilde{\gamma}_M + \tilde{\Gamma}_M} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle ds - M \int_{\tilde{\Gamma}_M} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle ds.$$

Since  $u_2$  is a subsolution and  $u_1$  is a solution of (2.1),

$$(5.17) \quad J_3 \leq \frac{\varepsilon}{4} M.$$

From (5.15), (5.16), (5.17) and the nonnegativity of  $J$ , we find

$$(5.18) \quad 0 \leq \int_{\gamma_M} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle ds + \frac{\varepsilon}{2} + 2|\Gamma| \frac{M_0}{M} + O(\varepsilon|\Delta|).$$

Consider now the domain  $\Delta'_M$  bounded by  $\gamma'$ ,  $\gamma_M$  and parts of  $\Gamma$ . Then

$$\begin{aligned} (5.19) \quad \int_{\gamma'} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle ds &\geq - \int_{-\gamma_M} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle ds \\ &\quad - \frac{\varepsilon}{2} - \varepsilon|\Delta'_M|. \end{aligned}$$

Combining (5.18) and (5.19) (remembering the difference in orientation of the common term) gives

$$(5.20) \quad \int_{\gamma'} \left\langle \frac{\nabla u_1}{W_1} - \frac{\nabla u_2}{W_2}, \nu \right\rangle ds \geq -\varepsilon - 2|\Gamma| \frac{M_0}{M} + O(\varepsilon|\Delta|).$$

We can now let  $M \rightarrow \infty$  and  $|\Delta| \rightarrow 0$  in (5.20) to obtain

$$F_u(\gamma') \geq F_w(\gamma') - \varepsilon.$$

Recalling (5.14) and letting  $\delta \rightarrow 0$  gives

$$(5.21) \quad F_u(\gamma') \geq |\gamma'| - K\varepsilon|\gamma'|^3 - \varepsilon.$$

Note that we cannot let  $\varepsilon$  tend to zero in (5.21) since  $|\gamma'| = \sqrt{\frac{\varepsilon}{K}} |\gamma|$ . Instead we divide our original arc  $\gamma$  into  $N = \sqrt{\frac{K}{\varepsilon}}$  small pieces  $\gamma'$ . Then summing (5.21) gives

$$F_u(\gamma) \geq |\gamma| - \varepsilon^2|\gamma|^3 - \sqrt{K\varepsilon}.$$

Since  $\varepsilon$  is arbitrary,  $F_u(\gamma) = |\gamma|$ .

The following lemma is a simple extension of Lemma 5.4.

LEMMA 5.5. *Let  $D$  be a domain bounded in part by an arc  $\gamma$  and let  $\{u_n\}$  be a sequence of solutions of (2.1) in  $D$  with each  $u_n$  continuous on  $\gamma$ . Then*

(i) *if the sequence tends to  $+\infty$  uniformly on compact subsets of  $\gamma$  while remaining uniformly bounded on compact subsets of  $D$ , we have*

$$\lim_{n \rightarrow \infty} \int_{\gamma} \left\langle \frac{\nabla u_n}{W_n}, \nu \right\rangle ds = |\gamma|.$$

(ii) *if the sequence tends to  $-\infty$  uniformly on compact subsets of  $\gamma$  while remaining uniformly bounded on compact subsets of  $D$ , we have*

$$\lim_{n \rightarrow \infty} \int_{\gamma} \left\langle \frac{\nabla u_n}{W_n}, \nu \right\rangle ds = -|\gamma|.$$

We also will need the following variant of Lemma 5.5.

LEMMA 5.6. *Let  $D$  be a domain bounded in part by an arc  $\gamma$  with  $\kappa(\gamma) = 2H$  and let  $\{u_n\}$  be a sequence of solutions of (2.1) in  $D$  with each  $u_n$  continuous on  $\gamma$ . Then if the sequence diverges to  $-\infty$  uniformly on compact subsets of  $D$  while*

remaining uniformly bounded on compact subsets of  $\gamma$ , we have

$$\lim_{n \rightarrow \infty} \int_{\gamma} \left\langle \frac{\nabla u_n}{W_n}, \nu \right\rangle ds = |\gamma|.$$

We shall omit the proof of Lemmas 5.5 and 5.6 which uses the technique of Lemma 5.4. See the proof of Lemma 4.5 of [9] for more details.

### 6. The local Harnack inequality and monotone convergence theorem.

Let  $u$  be a nonnegative or nonpositive solution of (2.1) in  $B_{\rho}(P)$  for  $\rho < R(P)$ , the injectivity radius at  $P$ . Then it follows from Theorem 1.1 of [10] that

$$(6.22) \quad |\nabla u(P)| \leq f\left(\frac{|u(P)|}{\rho}\right), \quad f(t) = e^{C(1+t)^2},$$

where  $C$  depends on  $\rho$  and the local geometry of  $\mathbb{M}$  but is independent of  $u$ . Following Serrin [8] we prove the following:

**THEOREM 6.1.** (Local Harnack inequality) *There exists a function  $\Phi(t, r)$  such that*

$$(6.23) \quad |u(Q)| \leq \Phi(m, r)$$

where  $m = |u(P)|$  and  $r = d(P, Q)$  the distance from  $P$  to  $Q$ . For each fixed  $t$ ,  $\Phi(t, r)$  is a continuous strictly increasing function defined on an interval  $0 \leq r < \rho(t)$  with

$$\Phi(t, 0) = t, \quad \Phi(t, r) \rightarrow \infty \text{ as } r \rightarrow \rho(t),$$

where  $\rho(t)$  is a continuous strictly decreasing function tending to zero as  $t$  tends to infinity.

*Proof.* Suppose  $u \geq 0$  (otherwise let  $v = -u$ ). Consider points  $Q$  lying on a fixed geodesic emanating from  $P$  and let  $u(r)$  denote the values of  $u$  along this ray. Then from (6.22),

$$\frac{du}{dr} \leq f\left(\frac{u(r)}{\rho - r}\right).$$

Define  $\Phi(t, r)$  by the conditions

$$(6.24) \quad \frac{d\Phi}{dr} = f\left(\frac{\Phi}{\rho - r}\right), \quad \Phi(t, 0) = t.$$

Then  $u(r) \leq \Phi(m, r)$  whenever  $\Phi$  is well defined. The solution of (6.24) is implicitly defined by  $\Phi = (\rho - r)v$  and

$$\int_{v_0}^v \frac{ds}{f(s) + s} = \log \frac{1}{\rho - r}, \quad v_0 = \frac{t}{\rho}.$$

Since the integral converges as  $v$  tends to infinity,  $\Phi$  is defined only on an interval  $0 \leq r < \rho(t)$  where clearly  $\rho(t) \rightarrow 0$  as  $t \rightarrow \infty$ . This completes the proof.  $\square$

The existence theorems of the following sections depend upon the limiting behavior of monotone increasing and monotone decreasing sequences of solutions in a fixed domain. As in the Euclidean case, we have the following consequence of Theorems 4.8 and 6.1.

**THEOREM 6.2.** (Monotone convergence theorem) *Let  $\{u_n\}$  be a monotonically increasing or decreasing sequence of solutions of (2.1) in a fixed domain  $D$ . If the sequence is bounded at a single point of  $D$ , there exists a nonempty open set  $U \subset D$  such that  $\{u_n\}$  converges to a solution in  $U$ . The convergence is uniform on compact subsets of  $U$  and the divergence is uniform on compact subsets of  $V = D \setminus U$ . If  $V$  is nonempty,  $\partial V$  consists of arcs of curvature  $\pm 2H$  and parts of  $\partial D$ . These arcs are convex to  $U$  for increasing sequences and concave to  $U$  for decreasing sequences.*

In particular, no component of  $V$  can consist of a single interior arc. We have more information about the set  $V$  if we have more knowledge about the boundary behavior of the sequence  $\{u_n\}$  and the curvature of  $\partial D$ .

**LEMMA 6.3.** *Let  $D$  be a domain bounded in part by an arc  $C$  with  $\kappa(C) \geq 2H$ . Let  $\{u_n\}$  be an increasing or decreasing sequence of solutions of (2.2) in  $D$  with each  $u_n$  continuous in  $D \cup C$ . Suppose  $\gamma$  is an interior arc of  $D$  of curvature  $2H$  forming part of the boundary of  $V$ . Then  $\gamma$  cannot terminate at an interior point of  $C$  if  $\{u_n\}$  either diverges on  $C$  to  $\pm\infty$  or remains uniformly bounded on compact subsets of  $C$ .*

*Proof.* Let  $\gamma$  be an arc in  $\partial V$  as in Lemma 6.3 that terminates at an interior point  $P$  of  $C$ . By considering only a small neighborhood of  $P$ , we may assume that  $C$  is  $C^2$ . By Theorem 4.8 the sequence  $\{u_n\}$  cannot diverge to  $-\infty$  on  $C$ . Moreover, if the curvature of  $C$  is not identically  $2H$  and  $\{u_n\}$  remains uniformly bounded on compact subsets of  $C$ , Lemma 4.9 insures that a neighborhood of  $C$  is contained in  $U$  or  $V$ , a contradiction. Hence assume  $\kappa(C) \equiv 2H$ . Suppose  $\{u_n\}$  diverges to  $+\infty$  on  $C$  and there exists exactly one such  $\gamma$  terminates at  $P$ . Let  $Q$  be a point of  $\gamma$  close to  $P$  and choose  $R$  on  $C$  close to  $P$  so that the geodesic segment  $\overline{RQ}$  lies in  $U$ . Let  $T$  be the triangle formed by  $\overline{RQ}$  and the constant curvature  $2H$  arcs  $\widehat{QP}$  and  $\widehat{PR}$ . Then by (5.11)

$$(6.25) \quad 2HA(T) = F_{u_n}(\widehat{QP}) + F_{u_n}(\widehat{PR}) + F_{u_n}(\overline{RQ})$$



while by Lemma 5.5

$$(6.26) \quad \lim_{n \rightarrow \infty} F_{u_n}(\widehat{QP}) = |\widehat{QP}|, \quad \lim_{n \rightarrow \infty} F_{u_n}(\widehat{PR}) = |\widehat{PR}|.$$

From (6.25), (6.26) and Lemma 5.3 we can conclude

$$(6.27) \quad \frac{2HA(T)}{|\widehat{RQ}|} \geq \frac{|\widehat{QP}| + |\widehat{PR}|}{|\widehat{RQ}|} - 1.$$

Keeping  $P$  fixed, we move  $Q$  to  $Q'$  and  $R$  to  $R'$  along the same arcs so that  $|\widehat{Q'P}| = \lambda |\widehat{QP}|$  and  $|\widehat{PR'}| = \lambda |\widehat{PR}|$  and form the triangle  $T'$  by joining  $Q'$  to  $R'$  by a geodesic. Then the left-hand side of (6.27) tends to zero as  $\lambda \rightarrow 0$  while the right-hand side of (6.27) remains uniformly positive, a contradiction. The only other possibility is that two arcs  $\gamma_1$  and  $\gamma_2$  terminate at  $P$ . Then again we can find a triangle  $T \subset U$  whose edges are two constant curvature  $2H$  arcs and a geodesic segment as before (perhaps with  $\partial T \cap C = \{P\}$ ). The same argument gives a contradiction.

In case the sequence remains uniformly bounded on compact subsets of  $C$  and there is exactly one  $\gamma$ , we choose  $R$  on  $C$  so that  $T$  is contained in  $V$ . By Lemma 4.4 the sequence must be divergent to  $-\infty$  in  $V$ . We now reach a contradiction as above by using Lemma 5.5. If there are two arcs terminating at  $P$ , then  $V$  is necessarily the convex lens domain formed by  $\gamma_1$  and  $\gamma_2$ . Choose the point  $Q$  on  $\gamma_1$  and  $R$  on  $\gamma_2$  and form  $T$  in  $V$ . Then (6.25), (6.26) and (6.27) still hold and we reach a contradiction as before.

**7. Existence and uniqueness.** In this section we prove existence theorems for constant mean curvature graphs in which the boundary values  $\pm\infty$  are allowed on entire arcs of the boundary. As we have seen in Theorem 4.8, these arcs must be of curvature  $\pm 2H$  with respect to the domain.

*Definition 7.1.* (Admissible domain) We say a bounded domain  $D$  is *admissible* if it is simply connected and  $\partial D$  is piecewise  $C^2$  and consists of three set of  $C^2$  open arcs  $\{A_i\}$ ,  $\{B_i\}$  and  $\{C_i\}$  satisfying  $\kappa(A_i) = 2H$ ,  $\kappa(B_i) = -2H$  and  $\kappa(C_i) \geq 2H$  respectively (with respect to the interior of  $D$ ). We suppose that no two of the arcs  $A_i$  and no two of the arcs  $B_i$  have a common endpoint. In addition if the family  $\{B_i\}$  is nonempty, we assume that the set  $D^*$  formed by replacing each arc  $B_i$  by  $B_i^*$ , the geodesic reflection of  $B_i$  across its endpoints, is a well-defined domain. Finally we assume that there is a bounded subsolution of (2.1) in  $D^*$  (recall this is automatic if  $\kappa(\partial D^*) \geq 2H$  or if  $D \subset \mathbb{H}^2$  and  $H \leq \frac{1}{2}$ ) so that by Theorem 3.2, the Dirichlet problem with finite data is well-posed in  $D^*$ .

*Remark 7.2.* Note that we do not insist that  $\kappa(P) \geq 2H$  for  $P$  an endpoint of the  $A_i$  or  $B_i$  so our definition allows admissible domains to be nonconvex.

*Definition 7.3.* (Dirichlet Problem) Given an admissible domain  $D$ , the generalized Dirichlet problem is to find a solution of (2.1) in  $D$  which assumes the value  $+\infty$  on each  $A_i$ ,  $-\infty$  on each  $B_i$  and assigned continuous data on each of the open arcs  $C_i$ . Note that the continuous data is allowed to become unbounded at the endpoints.

We shall need one more definition before we can start considering some special cases of the Dirichlet problem.

*Definition 7.4.* (Admissible polygon) Let  $D$  be an admissible domain. We say that  $\mathcal{P}$  is an admissible polygon if  $\mathcal{P}$  is a simple domain contained in  $\bar{D}$  with  $\partial\mathcal{P}$  piecewise smooth consisting of arcs of constant curvature  $\kappa = \pm 2H$  with vertices chosen from among the endpoints of the families  $\{A_i\}$ ,  $\{B_i\}$  and  $\{C_i\}$ . For an admissible  $\mathcal{P}$ , let  $\alpha$  and  $\beta$  be the total length of the arcs in  $\partial\mathcal{P}$  belonging to  $\{A_i\}$  and  $\{B_i\}$  respectively. Finally, let  $\ell$  be the perimeter of  $\mathcal{P}$  and  $\mathcal{A}(\mathcal{P})$  be the area of  $\mathcal{P}$ .

**PROPOSITION 7.5.** *Consider the Dirichlet problem in an admissible domain  $D$  and suppose the family  $\{B_i\}$  is empty,  $\kappa(C_i) > 2H$  and the assigned data  $f$  on the arcs  $C_i$  are bounded below. Then there exists a solution if and only if*

$$(7.28) \quad 2\alpha < \ell + 2H\mathcal{A}(\mathcal{P}) \text{ for all admissible polygons } \mathcal{P}.$$

*Proof.* Let  $u_n$  be the solution of (2.1) in  $D$  such that

$$u_n = \begin{cases} n & \text{on } \cup A_i \\ \min(n, f) & \text{on } \cup C_i. \end{cases}$$

Such a solution exists and is unique by Theorem 3.2. Moreover by the general maximum principle, the sequence  $\{u_n\}$  is monotone increasing so the monotone convergence Theorem 6.2 applies. Suppose  $V$  is nonempty. By Lemma 6.3 an interior arc of  $D$  which bounds  $V$  must be of curvature  $2H$  and can terminate only at an endpoint of some  $A_i$  or  $C_i$ . Moreover by Lemma 4.9, a neighborhood of each  $C_i$  is contained in  $U$ . Therefore the boundary of each component of  $V$  is an admissible polygon  $\mathcal{P}$  with vertices among those of the  $A_i$  and  $C_i$ . Also the curvature  $2H$  arcs forming the boundary which are not among the  $A_i$  are concave to  $V$  by Theorem 6.2. By Lemma 5.2 applied to each  $u_n$  in  $\mathcal{P}$ ,

$$(7.29) \quad 2H\mathcal{A}(\mathcal{P}) = F_n(\partial\mathcal{P} - \Sigma'A_i) + F_n(\Sigma'A_i),$$

where  $\Sigma'A_i$  is the union of the arcs  $A_i$  which are part of  $\mathcal{P}$ . Then by Lemma 5.5

$$(7.30) \quad \lim_{n \rightarrow \infty} F_n(\partial\mathcal{P} - \Sigma'A_i) = -(\ell - \alpha).$$

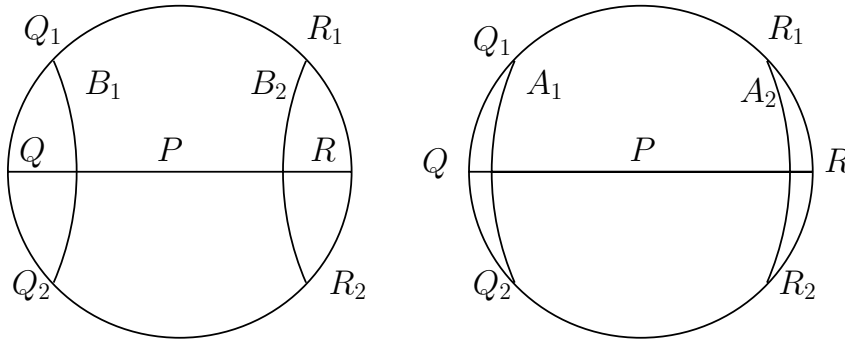


Figure 1. Barriers of Example 7.7.

But  $|F_n(\Sigma'A_i)| \leq \alpha$ , hence  $\ell - \alpha \leq \alpha - 2HA(\mathcal{P})$ , contradicting our assumption (7.28). Thus  $V$  is empty and the sequence converges uniformly on compact subsets of  $D$  to a solution  $u$ . Since each  $u_n$  is uniformly bounded in a neighborhood of each  $C_i$  by Lemma 4.9, a standard barrier argument shows that  $u = f$  on  $\cup C_i$ . Using (7.29) and Lemmas 5.3 and 5.4, the necessity of (7.28) is clear. This completes the proof.  $\square$

Similarly, we have:

PROPOSITION 7.6. *Consider the Dirichlet problem in an admissible domain  $D$  and suppose the family  $\{A_i\}$  is empty,  $\kappa(C_i) > 2H$  and the assigned data  $f$  on the arcs  $C_i$  are bounded above. Then there exists a solution if and only if*

$$(7.31) \quad 2\beta < \ell - 2HA(\mathcal{P}) \text{ for all admissible polygons } \mathcal{P}.$$

The proof is completely analogous using monotone decreasing sequences in  $D^*$ , which must diverge in a neighborhood of each  $B_i^*$  by Lemma 4.9. The simple case of one  $B$  and one  $C$  already illustrates the importance of considering the domain  $D^*$ , which is a key idea.

We now use Propositions 7.5 and 7.6 to construct some useful barriers and to remove the assumption  $\kappa(C_i) > 2H$ .

Example 7.7. Let  $B = B_\delta(P)$  be a ball of small radius  $\delta$  and let  $Q$  and  $R$  be “antipodal” points on  $\partial B$ . Choose points  $Q_1$  and  $Q_2$  on  $\partial B$  and symmetric with respect to the geodesic through  $QPR$ . Now let  $B_1$  be an arc of curvature  $-2H$  (as seen from  $P$ ) joining  $Q_1$  and  $Q_2$  and set  $A_1 = B_1^*$ . Let  $R_1$  and  $R_2$  on  $\partial B$  be reflections of  $Q_1$  and  $Q_2$  (with respect to the geodesic orthogonal to  $QPR$  through  $P$ ) and define  $B_2$  and  $A_2$ . Then for  $\delta$  small compared with  $H$ , the domain  $B^+$  bounded by  $A_1, A_2$  and parts of  $\partial B$  satisfies the conditions of Proposition 7.5 and similarly the domain  $B^-$  bounded by  $B_1, B_2$  and parts of  $\partial B$  satisfies the conditions of Proposition 7.6. Let  $u^+$  be the solution of (2.1) in  $D^+$  with boundary

values  $+\infty$  on  $A_1 \cup A_2$  and the constant value  $M$  on the remainder of the boundary. Similarly let  $u^-$  be the solution of (2.1) in  $D^-$  with boundary values  $-\infty$  on  $B_1 \cup B_2$  and the constant value  $-M$  on the remainder of the boundary.

Using these examples we prove:

PROPOSITION 7.8. *Let  $D$  be a domain bounded in part by an arc  $\gamma$  and let  $\{u_n\}$  be a sequence of solutions of (2.1) in  $D$  which converge uniformly on compact subsets of  $D$  to a solution  $u$ . Suppose each  $u_n$  is continuous in  $D \cup \gamma$ . Then*

(i) *Suppose the boundary values of  $u_n$  converge uniformly on compact subset of  $\gamma$  to a bounded limit  $f$ . If  $\kappa(\gamma) \geq 2H$ , then  $u$  is continuous in  $D \cup \gamma$  and  $u = f$  on  $\gamma$ .*

(ii) *If  $\kappa(\gamma) = 2H$  and the boundary values of  $u_n$  diverge uniformly to  $+\infty$  on compact subsets of  $\gamma$ , then  $u$  takes on the boundary values  $+\infty$  on  $\gamma$ .*

(iii) *If  $\kappa(\gamma) = -2H$  and the boundary values of  $u_n$  diverge uniformly to  $-\infty$  on compact subsets of  $\gamma$ , then  $u$  takes on the boundary values  $-\infty$  on  $\gamma$ .*

*Proof.* (i) It suffices to prove that the sequence  $\{u_n\}$  is uniformly bounded in the intersection of  $D$  with a neighborhood of any interior point  $P$  of  $\gamma$ . Orient the ball  $B$  of Example 7.7 so that geodesic joining  $QPR$  is tangent to  $\partial D$ . We may choose the points  $Q_i$   $i = 1, 2$  and  $\delta$  small so that the boundary arc joining  $Q_2$  and  $R_2$  lies in a compact subset of  $D$ . Then if  $M$  is large enough,

$$u_n \leq u^+ \text{ in } D \cap B^+ \quad \text{and} \quad u_n \geq u^- \text{ in } D \cap B^-.$$

Therefore the sequence is uniformly bounded in a neighborhood of  $P$ .

(ii) Let  $P$  be an interior point of  $\gamma$ . Similarly as in (i), we obtain that there exists  $M$  large enough so that  $u_n \geq -M$  in  $N = B_\epsilon(P) \cap D$ . Let  $v_m$  be the solution of (2.1) in  $N$  with boundary values  $m$  on  $\gamma \cap B_\epsilon(P)$  and  $-M$  on the remaining boundary. By the general maximum principle  $u_n \geq v_m$  for  $n$  sufficiently large so  $u \geq v_m$  in  $N$ . In particular,  $u(P) > m$  for every  $m$  and  $u$  must take on the value  $+\infty$  at  $P$ .

(iii) Again for  $P$  interior to  $\gamma$ ,  $u_n \leq M$  in  $N = B_\epsilon(P) \cap D$ . Let  $v_m$  be the solution of (2.1) in  $N$  with boundary values  $-m$  on  $\gamma \cap B_\epsilon(P)$  and  $M$  on the remaining boundary. By the general maximum principle  $u_n \leq v_m$  for  $n$  sufficiently large so  $u \leq v_m$  in  $N$ . Since the  $v_m$  are monotonically decreasing (and converges to a solution with infinite boundary values on  $\gamma \cap B_\epsilon(P)$ ),  $u$  must take on the value  $-\infty$  at  $P$ . □

We can now extend Propositions 7.5 and 7.6 to allow the arcs  $C_i$  to satisfy  $\kappa(C_i) \geq 2H$ . The only change needed in the proof of Proposition 7.5 is to use part (i) of Proposition 7.8 to show that the solution takes on the required boundary data on the arcs  $C_i$ . The extension of Proposition 7.6 is more delicate since if  $\kappa(C_i) = 2H$  for some  $i$ , we do not know that the sequence is bounded below in a neighborhood of  $C_i$ . However by Lemma 6.3, a neighborhood of  $C_i$  is either

contained in  $U$  or  $V$ . We have already handled the former case. In the latter case, consider a component of  $V$  whose boundary is an admissible polygon  $\mathcal{P}$  (with vertices among those of the  $B_i$  and  $C_i$ ) containing a subset  $\Sigma' C_i$  of the  $C_i$  of curvature  $2H$  and a subset  $\Sigma' B_i$  of the  $B_i$ . The interior arcs of  $D$  which are in  $\mathcal{P}$  are convex to  $V$ . By Lemma 5.2 applied to each  $u_n$  in  $\mathcal{P}$ ,

$$(7.32) \quad 2HA(\mathcal{P}) = F_n(\Sigma' B_i) + F_n(\Sigma' C_i) + F_n(\partial\mathcal{P} - \Sigma' B_i - \Sigma' C_i).$$

Then by Lemma 5.5

$$(7.33) \quad \lim_{n \rightarrow \infty} F_n(\partial\mathcal{P} - \Sigma' B_i - \Sigma' C_i) = \ell - \beta - \Sigma' |C_i|,$$

and by Lemma 5.6

$$(7.34) \quad \lim_{n \rightarrow \infty} F_n(\Sigma' C_i) = \Sigma' |C_i|.$$

But  $|F_n(\Sigma' B_i)| \leq \beta$ , hence

$$2HA(\mathcal{P}) \geq -\beta + \Sigma' |C_i| + (\ell - \Sigma' |C_i| - \beta) = \ell - 2\beta,$$

contradicting our assumption (7.31). Thus  $V$  is empty and the sequence converges uniformly on compact subsets of  $D$  to a solution  $u$ . Finally, we use parts (i) and (iii) of Proposition 7.8 to show that our solution achieves the boundary values. We state these results as:

**THEOREM 7.9.** *Consider the Dirichlet problem in an admissible domain  $D$  and suppose the family  $\{B_i\}$  is empty and the assigned data  $f$  on the arcs  $C_i$  are bounded below. Then there exists a solution if and only if*

$$(7.35) \quad 2\alpha < \ell + 2HA(\mathcal{P}) \text{ for all admissible polygons } \mathcal{P}.$$

**THEOREM 7.10.** *Consider the Dirichlet problem in an admissible domain  $D$  and suppose the family  $\{A_i\}$  is empty and the assigned data  $f$  on the arcs  $C_i$  are bounded above. Then there exists a solution if and only if*

$$(7.36) \quad 2\beta < \ell - 2HA(\mathcal{P}) \text{ for all admissible polygons } \mathcal{P}.$$

In the following theorem we allow both families  $\{A_i\}$  and  $\{B_i\}$  to occur and allow the data  $f$  on the  $\{C_i\}$  to be unbounded both above and below as we approach the endpoints.

**THEOREM 7.11.** *Consider the Dirichlet problem in an admissible domain  $D$  and suppose the family  $\{C_i\}$  is nonempty. Then there exists a solution if and only if*

$$(7.37) \quad 2\alpha < \ell + 2HA(\mathcal{P}) \quad \text{and} \quad 2\beta < \ell - 2HA(\mathcal{P})$$

*for all admissible polygons  $\mathcal{P}$ .*

*Proof.* By Theorem 7.9 the first condition of (7.37) guarantees the existence of a solution  $u^+$  of (2.1) in  $D^*$  such that

$$u^+ = \begin{cases} +\infty & \text{on } \cup A_i \\ 0 & \text{on } \cup B_i^* \\ \max(f, 0) & \text{on } \cup C_i. \end{cases}$$

Similarly by Theorem 7.10, the second condition of (7.37) guarantees the existence of a solution  $u^-$  of (2.1) in  $D^*$  such that

$$u^- = \begin{cases} -\infty & \text{on } \cup B_i \\ 0 & \text{on } \cup A_i \\ \min(f, 0) & \text{on } \cup C_i. \end{cases}$$

Now let  $u_n$  be the solution of (2.1) in  $D^*$  such that

$$u_n = \begin{cases} n & \text{on } \cup A_i \\ -n & \text{on } \cup B_i^* \\ f_n & \text{on } \cup C_i \end{cases}$$

where  $f_n$  is the truncation of  $f$  above by  $n$  and below by  $-n$ .

By the general maximum principle,

$$u_n \leq u^+ \text{ in } D^* \quad \text{and} \quad u^- \leq u_n \text{ in } D.$$

Therefore the sequence  $\{u_n\}$  is uniformly bounded on compact subsets of  $D$  and so a subsequence converges uniformly on compact subsets to a solution  $u$  in  $D$ . By Proposition 7.8,  $u$  takes on the boundary assigned boundary data. The necessity of the conditions (7.37) follows essentially as in the Theorems 7.9 and 7.10.  $\square$

We turn next to the important case when the family  $\{C_i\}$  is empty.

**THEOREM 7.12.** *Consider the Dirichlet problem in an admissible domain  $D$  and suppose the family  $\{C_i\}$  is empty. Then there exists a solution if and only if*

$$(7.38) \quad \alpha = \beta + 2HA(D)$$

and

$$(7.39) \quad 2\alpha < \ell + 2HA(\mathcal{P}) \quad \text{and} \quad 2\beta < \ell - 2HA(\mathcal{P})$$

for all other admissible polygons  $\mathcal{P}$ .

*Proof.* Let  $v_n$  be the solution of (2.1) in  $D^*$  with boundary values  $n$  on each  $A_i$  and 0 on each  $B_i^*$ . For  $0 < c < n$  we define for  $n \geq 1$

$$E_c = \{v_n - v_0 > c\} \quad \text{and} \quad F_c = \{v_n - v_0 < c\};$$

we suppress the dependence of these sets on  $n$ . Let  $E_c^i$  and  $F_c^i$  denote respectively the components of  $E_c$  and  $F_c$  whose closure contains respectively  $A_i$  and  $B_i^*$ . By the general maximum principle,  $E_c = \cup E_c^i$  and  $F_c = \cup F_c^i$ . If  $c$  is sufficiently close to  $n$ , the sets  $\{E_c^i\}$  will be distinct and disjoint (to see this, note that we can separate any two of the  $A_i$  by a curve joining two of the  $B_i^*$  on which  $v_n - v_0$  is bounded away from  $n$ ). Now define  $\mu(n)$  to be the infimum of the constants  $c$  such that the sets  $\{E_c^i\}$  are distinct and disjoint. The sets  $\{E_\mu^i\}$  will again be distinct although there must be at least one pair  $(i, j)$ ,  $i \neq j$  such that  $\bar{E}_\mu^i \cap \bar{E}_\mu^j$  is nonempty. This implies that given any  $F_\mu^i$  there is some  $F_\mu^j$  distinct from it. Now let  $u_i^+$ ,  $i = 1, \dots, k$  be the solution of (2.1) in  $D^*$  taking on the boundary values  $+\infty$  on  $A_i$  and 0 on the remaining boundary. This solution exist by Theorem 7.9 since the solvability condition (7.35) follows trivially from (7.38), (7.39). Also let  $u_i^-$  be the solution of (2.1) in the domain  $\tilde{D}$  bounded by  $\cup A_i, B_i^*, \cup_{j \neq i} B_j$  taking on the boundary values  $-\infty$  on  $\cup_{j \neq i} B_j$  and 0 on the remainder of the boundary. We claim this solution exists by Theorem 7.10. To verify (7.36), we need only consider admissible polygons  $\tilde{\mathcal{P}}$  in  $\tilde{D}$  which contain the lens domain  $L$  formed by  $B_j$  and  $B_i^*$ . Let  $\mathcal{P}$  be the corresponding admissible polygon for  $D$  formed by deleting  $L$ . By (7.38), (7.39) we have

$$2\beta = 2(|B_i| + \sum_{j \neq i} |B_j|) \leq \ell - 2HA(\mathcal{P}),$$

or equivalently

$$2\tilde{\beta} = 2\sum_{j \neq i} |B_j| \leq \tilde{\ell} - 2HA(\tilde{\mathcal{P}}) + (2HA(L) - 2|B_i|).$$

However since a solution (for example  $v_n$ ) exists in  $L$ , we have by Lemma 5.3  $2HA(L) < 2|B_i|$  so condition (7.36) is satisfied.  $\square$

We now set

$$u^+(p) = \max_i \{u_i^+(p)\} \quad \text{in } D^* \quad \text{and} \quad u^-(p) = \min_i \{u_i^-(p)\} \quad \text{in } D.$$

We note that if we compare each  $u_i^+$  to a fixed bounded subsolution in  $D^*$  (which exists since  $D$  is admissible), then by the general maximum principle there is a constant  $N > 0$  such that  $u_i^+ > -N$ ,  $i = 1, \dots, k$ . Finally we set  $u_n = v_n - \mu(n)$ .

We now claim that

$$u_n \leq u^+ + M \quad \text{in } D^* \quad \text{and} \quad u_n \geq u^- - M \quad \text{in } D.$$

where  $M = N + \sup_{D^*} |\nu_0|$ . Suppose  $u_n > \nu_0$  at some point  $P$ . Then  $u_n - \nu_0 > \mu(n)$  at  $P$  so that  $P$  is in some  $E_\mu^i$ . Applying the general maximum principle in the domain  $E_\mu^i$ , we obtain

$$u_n \leq u_i^+ + N + \sup_{E_\mu^i} |\nu_0| \leq u^+ + M \quad \text{at } P.$$

On the other hand, suppose  $u_n < \nu_0$  at some point  $P \in D$ . Then  $\nu_0 - u_n > \mu(n)$  at  $P$  so that  $P$  is in some  $F_\mu^i$ . By what has been shown above, there is a corresponding  $j = j(i)$  so that  $F_\mu^i \cap F_\mu^j = \emptyset$ . Applying the general maximum principle in  $F_\mu^i$ , we obtain

$$u_n \geq u_j^- - \sup_{F_\mu^i} |\nu_0| \geq u^- - M.$$

Therefore the claim is justified and the sequence  $\{u_n\}$  is uniformly bounded on compact subsets of  $D$ . By the compactness principle, a subsequence  $\{u_n\}$  converges uniformly on compact subsets of  $D$  to a solution  $u$ . We still must show that  $u$  takes on the required boundary values. We observe that a subsequence  $\mu(n)$  diverges to  $+\infty$  otherwise we can extract a subsequence converging to a finite limit  $\mu_0$ . Each  $u_n$  would then be bounded below in  $D^*$  uniformly in  $n$ , and the boundary values of  $u_n$  would tend uniformly on compact subsets of  $\cup B_i^*$  to  $-\mu_0$  and diverge uniformly to  $+\infty$  on  $\cup A_i$ . Once again we could find a subsequence converging uniformly on compact subsets to a solution in  $D^*$ . By Proposition 7.8 we would have

$$v = \begin{cases} +\infty & \text{on } \cup A_i \\ -\mu_0 & \text{on } \cup B_i^*. \end{cases}$$

We can now obtain a contradiction to (7.38) by a flux argument. By Lemma 5.2,

$$(7.40) \quad 2HA(D) = F_v(\Sigma A_i) + F_v(\Sigma B_i),$$

while by Lemmas 5.3 and 5.4

$$(7.41) \quad |F_v(\Sigma B_i)| < \beta \quad \text{and} \quad F_v(\Sigma A_i) = \alpha.$$

Combining (7.40) and (7.41) gives  $\alpha - \beta < 2HA(D)$ , a contradiction. In the same way, we see  $n - \mu(n)$  diverges to  $+\infty$ . Summing up we have shown that the boundary values of  $u_n$ , namely,  $-\mu(n)$  on  $\cup B_i^*$  and  $n - \mu(n)$  on  $\cup A_i$  diverge to  $-\infty$  and  $+\infty$  respectively. A now familiar argument shows that  $u_n$  diverges to  $-\infty$  on  $\cup B_i$ . The necessity of the conditions (7.38) and (7.39) is straightforward. The theorem is proved.



We end the section with a maximum principle that is valid for solutions with infinite boundary values. This result immediately proves uniqueness for the Dirichlet problem in an admissible domain if the family  $\{C_i\}$  is nonempty and uniqueness up to translation if the family  $\{C_i\}$  is empty.

**THEOREM 7.13.** *Let  $D$  be a domain whose boundary contains two families  $\{A_i\}$  and  $\{B_i\}$  of arcs satisfying  $\kappa(A_i) = 2H$  and  $\kappa(B_i) = -2H$ . Let  $E$  be a finite set of exceptional points on  $\partial D$  and let  $C$  denote the set of points in  $\partial D \setminus E$  which are not in the closure of the families  $\{A_i\}$  and  $\{B_i\}$ . Let  $u^1$  and  $u^2$  be solutions of (2.1) in  $D$  taking on the values  $+\infty$  on each  $A_i$  and  $-\infty$  on each  $B_i$ . If  $C$  is nonempty, assume  $\limsup (u^1 - u^2) \leq 0$  for any approach to points of  $C$ . If  $C$  is empty, assume  $u^1 \leq u^2$  at some point  $P$  in  $D$ . Then in either case  $u^1 \leq u^2$  in  $D$ .*

*Proof.* For simplicity we give the proof in case  $E$  is empty; the general case requires only a slight modification. As in the proof of the general maximum principle, let  $N$  and  $\varepsilon$  be positive constants with  $N$  large and  $\varepsilon$  small. Define

$$\varphi = \begin{cases} N - \varepsilon & \text{if } u^1 - u^2 \geq N \\ u^1 - u^2 - \varepsilon & \text{if } \varepsilon < u^1 - u^2 < N \\ 0 & \text{if } u^1 - u^2 \leq \varepsilon. \end{cases}$$

Then  $\varphi$  is Lipschitz and vanishes in a neighborhood of any point of  $C$ . Moreover,  $0 \leq \varphi < N$  and  $\nabla \varphi = \nabla u^1 - \nabla u^2$  in the set where  $\varepsilon < u^1 - u^2 < N$  and  $\nabla \varphi = 0$  almost everywhere in the complement of this set. Let  $D_{\varepsilon, \delta}$  denote the set of points in  $D$  whose distance from  $\partial D$  is at least  $\delta$  and whose distance from the endpoints of  $\{A_i\}$  and  $\{B_i\}$  is at least  $\varepsilon$ , where  $\delta < \varepsilon$ . For small  $\delta$ ,  $\Gamma := \partial D_{\varepsilon, \delta}$  consists of arcs  $A'_i$  parallel to  $A_i$ ,  $B'_i$  parallel to  $B_i$ ,  $C'$  parallel to  $C$  (all at distance  $\delta$ ) and small spherical arcs of radius  $\varepsilon$  with centers at the endpoints of the  $A_i$  and  $B_i$ . Now let

$$J = \int_{\Gamma} \varphi \left\langle \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2}, \nu \right\rangle ds,$$

where  $\nu$  is the outer normal to  $\Gamma$  with respect to the domain  $D_{\varepsilon, \delta}$ . We can estimate  $J$  from above in two ways:

$$(7.42) \quad \begin{aligned} J &= \int_{\Gamma} \varphi \left( 1 - \left\langle \frac{\nabla u^2}{W_2}, \nu \right\rangle \right) ds - \int_{\Gamma} \varphi \left( 1 - \left\langle \frac{\nabla u^1}{W_1}, \nu \right\rangle \right) ds \\ &\leq M \int_{\Gamma \setminus C'} \left( 1 - \left\langle \frac{\nabla u^2}{W_2}, \nu \right\rangle \right) ds \end{aligned}$$

or

$$(7.43) \quad J = \int_{\Gamma} \varphi \left( 1 + \left\langle \frac{\nabla u^1}{W_1}, \nu \right\rangle \right) ds - \int_{\Gamma} \varphi \left( 1 + \left\langle \frac{\nabla u^2}{W_2}, \nu \right\rangle \right) ds \\ \leq M \int_{\Gamma \setminus C'} \left( 1 + \left\langle \frac{\nabla u^1}{W_1}, \nu \right\rangle \right) ds,$$

since for small enough  $\delta$ ,  $\varphi \equiv 0$  in a neighborhood of  $C'$ . Now applying Lemmas 5.2 and 5.4 in the domain bounded by  $A_i, A'_i$  and the two spherical arcs of radius  $\varepsilon$  gives

$$(7.44) \quad \int_{A'_i} \left( 1 - \left\langle \frac{\nabla u^j}{W_j}, \nu \right\rangle \right) ds = O(\varepsilon), \quad j = 1, 2.$$

Similarly considering the domain bounded by  $B_i, B'_i$  and the two spherical arcs of radius  $\varepsilon$  gives

$$(7.45) \quad \int_{B'_i} \left( 1 + \left\langle \frac{\nabla u^j}{W_j}, \nu \right\rangle \right) ds = O(\varepsilon), \quad j = 1, 2,$$

where  $\nu$  is the outer normal to  $B'_i$  with respect to the domain  $D_{\varepsilon, \delta}$ . Using (7.42), (7.43), (7.44) and (7.45) we find that  $J = O(N\varepsilon)$ . On the other hand by Lemma 2.1

$$(7.46) \quad J = \int_{D_{\varepsilon, \delta}} \left\langle \nabla \varphi, \frac{\nabla u^1}{W_1} - \frac{\nabla u^2}{W_2} \right\rangle dV \geq 0.$$

Now consider a sequence of values of  $\varepsilon$  decreasing to zero with corresponding values  $\delta(\varepsilon)$  also decreasing to zero. Then  $D_{\varepsilon, \delta}$  is increasing so that  $J$  is monotone increasing (recall Lemma 2.1). It follows that  $\nabla u^1 \equiv \nabla u^2$  in the set where  $0 < u^1 - u^2 < N$ . Since  $N$  is arbitrary,  $\nabla u^1 \equiv \nabla u^2$  in the set where  $u^1 > u^2$ . It follows that if the set  $u^1 > u^2$  contains any nontrivial component, then  $u^1 \equiv u^2 + \text{positive constant}$  in that component and so by the maximum principle,  $u^1 \equiv u^2 + \text{positive constant}$  in  $D$ . By the assumptions of the theorem, the constant must be nonpositive, a contradiction. Thus  $u^1 \leq u^2$  in  $D$ . The same proof holds if  $C$  is empty.

**8. Examples of Scherk type.** Suppose  $\mathbb{M} = \mathbb{H}^2$  and  $H \leq \frac{1}{2}$ . We now construct polygons with an arbitrary number of sides that are admissible domains satisfying the Jenkins-Serrin conditions. Let  $S_\theta$  be the sector of angle  $\theta$  in a Euclidean circle of radius  $b$  and center  $O$  and denote by  $T_\alpha$  the triangle  $(O, C_1, C_2)$ , where the angle at  $O$  is  $\alpha$  and the points  $C_1, C_2$  are on the circle. Let  $I$  be the midpoint of  $[C_1, C_2]$  and let  $2d_\alpha := \text{dist}(C_1, C_2)$  and  $c_\alpha := \text{dist}(O, I)$ , see Figure 2.

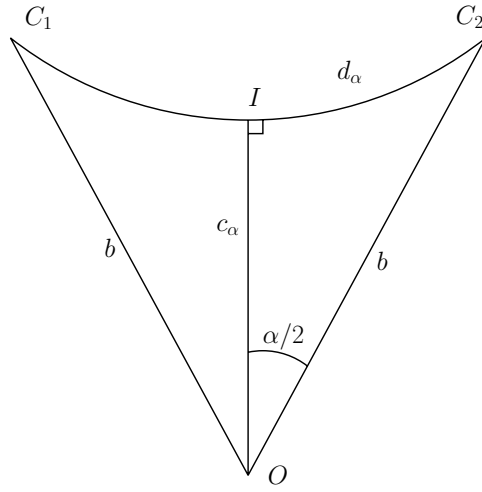


Figure 2. Triangle  $T(\alpha)$ .

We define a lens  $L_H(2d)$ , to be a region bounded by a geodesic of length  $2d$  and a segment of curvature  $\kappa = 2H$ . When the geodesic is fixed, there are two lenses with this boundary. By  $T(\alpha) + L_H(2d_\alpha)$  we mean  $T(\alpha)$  union the lens  $L_H(2d_\alpha)$  whose intersection with  $T(\alpha)$  is the geodesic  $[C_1, C_2]$ . Also  $T(\alpha) - L_H(2d_\alpha)$  is the complement in  $T(\alpha)$  of the lens  $L_H(2d_\alpha)$  having  $[C_1, C_2]$  in its boundary and intersecting  $T(\alpha)$  in a region; (cf. Figure 3).

We consider the domain  $D(\alpha, \beta) \subset S_{\alpha+\beta}$  defined as (see Figure 4):

$$D(\alpha, \beta) = (T(\alpha) + L_H(2d_\alpha)) \cup (T(\beta) - L_H(2d_\beta)).$$

The boundary  $\partial D(\alpha, \beta)$  consists of two geodesics of length  $b$ , and two arcs  $A, B$  of constant curvature  $\kappa = 2H$ , with curvature vectors pointing inside  $D(\alpha, \beta)$  on  $A$  and outside  $D(\alpha, \beta)$  on  $B$ .

PROPOSITION 8.1. *Let  $\theta \in [0, \pi]$ . There exists  $S_\alpha \cup S_\beta = S_\theta$ , a partition of  $S_\theta$  such that:*

$$|A| - |B| = 2H|D(\alpha, \beta)|$$

where  $|D(\alpha, \beta)| = \text{Area}(D(\alpha, \beta))$ . The perimeter  $|\partial D(\alpha, \beta)| = 2b + |A| + |B|$ .

*Proof.* In this proof we will use several formulas of hyperbolic geometry which we derive in the appendix. We consider the function

$$F(\alpha, \beta) = 2H|D(\alpha, \beta)| + |B| - |A|.$$

By Lemma 9.3 (here we are using  $H \leq \frac{1}{2}$ ), we have  $F(t) = F(t\theta, (1-t)\theta)$  is a continuous function on  $t \in [0, 1]$  with  $F(0) = F(0, \theta) > 0$  and  $F(1) = F(\theta, 0) < 0$ . □

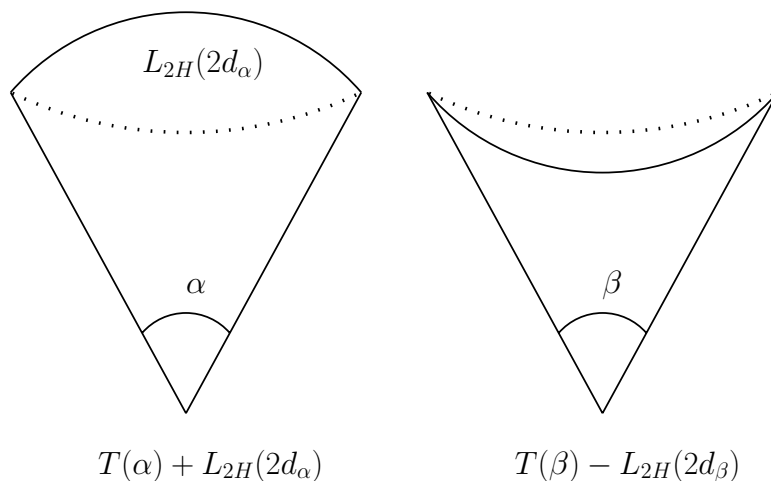


Figure 3. Domain  $D(\alpha, 0)$  et  $D(0, \beta)$ .

Fix an integer  $n$ , and apply Proposition 8.1 to the sector  $S_{\frac{2\pi}{n}}$ . We do  $n$  consecutive reflections through the geodesic sides of  $S_{\frac{2\pi}{n}}$  to obtain a simple closed curve  $\partial D(n)$  composed of  $n$  arcs of constant curvature  $\kappa = +2H$  and  $n$  arcs with curvature  $\kappa = -2H$ . Since the Euclidean circle centered at  $O$  of radius  $b$  has curvature greater than one, such domains  $D$  are inscribed in this circle. Here the sign of  $\kappa$  is with respect to the disk  $D(n)$  bounded by  $\partial D(n)$ . It is easy to see that  $D(n)$  is an admissible domain (see definition 7.1) with only boundary components of type  $A_i$  and  $B_i$ . Thus to solve the Dirichlet problem with boundary data  $+\infty$  on the  $A_i$  and  $-\infty$  on the  $B_i$  of  $\partial D(n)$ , it suffices to show the conditions (7.38) and (7.39) of Theorem 7.12 are satisfied.

Now we denote  $D(n)$  by  $D$  and  $\partial D(n)$  by  $\partial D$ . So we must show that for  $\mathcal{P}$  an admissible polygon inscribed in  $D$  we have

$$(7.38) \quad n|A| = n|B| + 2HA(D), \text{ for } \mathcal{P} = D,$$

and

$$(7.39a) \quad 2\lambda|A| < |\partial\mathcal{P}| + 2HA(\mathcal{P})$$

$$(7.39b) \quad 2\mu|B| < |\partial\mathcal{P}| - 2HA(\mathcal{P})$$

for  $\mathcal{P} \neq D$  and  $\lambda, \mu$  the number of  $A_i$  and  $B_i$ 's sides in  $\partial\mathcal{P}$  respectively. The condition (7.38) is satisfied for  $\mathcal{P} = D$  by construction: the perimeter of  $\mathcal{P}$  is given by  $|\partial D| = n|A| + n|B|$  and (by Proposition 8.1)

$$n(|A| - |B|) = 2HA(D).$$

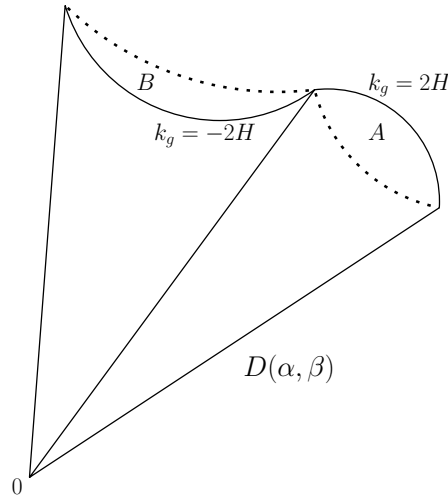


Figure 4. Domain  $D(\alpha, \beta)$ .

Now let  $\mathcal{P}$  an admissible polygon in  $D$ ,  $\mathcal{P} \neq D$ .

*Definition 8.2.* An elementary admissible polygon  $\mathcal{P}$  is an admissible polygon contained in a half-disk such that  $(D - \mathcal{P})$  have only one connected component.

We first check that the Jenkins-Serrin inequalities (7.39) are satisfied for elementary polygons. The elementary polygons have only one additional side  $C$ , the boundary of a lens which can be concave or convex with respect to  $\mathcal{P}$ . When it is convex (resp. concave) we denote the admissible polygon by  $\mathcal{P}_i$  (resp.  $\mathcal{P}_e$ ). We have three cases, depending on  $(\lambda, \mu) = (k, k), (k + 1, k)$  or  $(k, k + 1)$  with  $k$  chosen so that  $\lambda\alpha + \mu\beta \leq \pi$ . The perimeter is given by  $|\partial\mathcal{P}| = \lambda|A| + \mu|B| + |C|$ .

We remark that  $|C|$  is the length of an equidistant curve which bound a lens associated to a geodesic of length  $2d_{\lambda\alpha + \mu\beta}$ . The relationship between the area is  $\mathcal{A}(\mathcal{P}_i(\lambda, \mu)) = \mathcal{A}(\mathcal{P}_e(\lambda, \mu)) + 2\mathcal{A}(L_H(2d_{\lambda\alpha + \mu\beta}))$ . Thus it suffices to prove:

$$2\lambda|A| - |\partial\mathcal{P}(\lambda, \mu)| < 2H\mathcal{A}(\mathcal{P}_e(\lambda, \mu)) < 2H\mathcal{A}(\mathcal{P}_i(\lambda, \mu)) < |\partial\mathcal{P}(\lambda, \mu)| - 2\mu|B|.$$

We have

$$\begin{aligned} \mathcal{A}(\mathcal{P}_i(\lambda, \mu)) &= k|D(\alpha, \beta)| + (\lambda - k)|D(\alpha, 0)| + (\mu - k)|D(0, \beta)| \\ &\quad - |T(\lambda\alpha + \mu\beta)| + |L_H(2d_{\lambda\alpha + \mu\beta})|. \end{aligned}$$

By definition of  $F(\alpha, \beta)$  and Lemma 9.3 of the appendix we have

$$\begin{aligned} 2H\mathcal{A}(\mathcal{P}_i(\lambda, \mu)) - |\partial\mathcal{P}(\lambda, \mu)| + 2\mu|B| &= (\lambda - k)F(\alpha, 0) + (\mu - k)F(0, \beta) \\ &\quad - F(0, \lambda\alpha + \mu\beta) < 0. \end{aligned}$$

To check the sign, we remark  $F(\alpha, 0) < 0$ ,  $F(0, \lambda\alpha + \mu\beta) > 0$  and since  $d_\alpha < d_{\lambda\alpha + \mu\beta}$ , we use  $|A| < |C|$ :

$$\begin{aligned} F(0, \beta) - F(0, \lambda\alpha + \mu\beta) &= -F(\alpha, 0) - F(0, \lambda\alpha + \mu\beta) \\ &< (1 - 4H^2)(|A| - |C|) + 4H \arcsin(2H \tanh d_\alpha) \\ &\quad - 4H \arcsin(2H \tanh d_{\lambda\alpha + \mu\beta}) < 0. \end{aligned}$$

Similarly we have

$$\begin{aligned} 2HA(\mathcal{P}_e(\lambda, \mu)) + |\partial\mathcal{P}(\lambda, \mu)| - 2\lambda|A| &= (\lambda - k)F(\alpha, 0) + (\mu - k)F(0, \beta) \\ &\quad - F(\lambda\alpha + \mu\beta, 0) > 0. \end{aligned}$$

To check the sign, we remark  $F(0, \beta) > 0$ ,  $F(\lambda\alpha + \mu\beta, 0) < 0$  and  $F(\alpha, 0) - F(\lambda\alpha + \mu\beta, 0) > 0$  by Lemma 9.3.

Now let  $\mathcal{P}$  be an admissible polygon. We can write  $\mathcal{P} = D - \sum \mathcal{P}_i$  where each  $\mathcal{P}_i$  is an elementary admissible polygon or its complement in  $D$ . To conclude we prove the following lemma:

**LEMMA 8.3.** *We consider  $\mathcal{P}$  an admissible polygon with  $\mathcal{P} = D - \sum \mathcal{P}_i$  for a finite collection of admissible polygons  $\mathcal{P}_i$  satisfying  $|\partial\mathcal{P}_i| = \lambda_i|A| + \mu_i|B| + |C_i|$*

$$2\lambda_i|A| - |\partial\mathcal{P}_i| < 2HA(\mathcal{P}_i) < |\partial\mathcal{P}_i| - 2\mu_i|B|.$$

Then

$$2\lambda|A| - |\partial\mathcal{P}| < 2HA(\mathcal{P}) < |\partial\mathcal{P}| - 2\mu|B|.$$

*Proof.* We have  $|\partial\mathcal{P}| = (n - \sum \lambda_i)|A| + (n - \sum \mu_i)|B| + \sum |C_i|$ . Using  $2HA(\mathcal{P}) = n(|A| - |B|)$  we have

$$n(|A| - |B|) + \sum 2\mu_i|B| - \sum |\partial\mathcal{P}_i| < 2HA(\mathcal{P}) - \sum 2HA(\mathcal{P}_i)$$

$$2HA(\mathcal{P}) - \sum 2HA(\mathcal{P}_i) < n(|A| - |B|) - \sum 2\lambda_i|A| + \sum |\partial\mathcal{P}_i|$$

which gives the result by observing  $\lambda = n - \sum \lambda_i$  and  $\mu = n - \sum \mu_i$ .  $\square$

## 9. Appendix.

### 9.1. Hyperbolic geometry.

**LEMMA 9.1.** *In a Euclidean circle of radius  $R = \tanh \frac{b}{2}$ , centered at the origin  $O$  in the disk model of  $\mathbb{H}^2$ , we consider a triangle  $(O, C_1, C_2)$  where the angle at the center is  $\alpha$  and  $\text{dist}(O, C_1) = \text{dist}(O, C_2) = b$ . The point  $I$  is the mid-point*

of  $[C_1, C_2]$ . With  $b$  fixed, we note  $\text{dist}(O, I) = c_\alpha$ ,  $\text{dist}(C_1, I) = d_\alpha$  and the area  $|T(\alpha)| = \text{Area}(O, C_1, C_2)$ .

- (1) (Pythagoras)  $\cosh b = \cosh d_\alpha \cosh c_\alpha$ .
- (2)  $\sinh d_\alpha = \sinh b \sin(\alpha/2)$  and  $\tanh c_\alpha = \tanh b \cos(\alpha/2)$ .
- (3)  $\sin \frac{|T(\alpha)|}{2} = \frac{\sinh c_\alpha \sinh d_\alpha}{1 + \cosh b}$ .

LEMMA 9.2. *We consider the half-lens  $L_H(2d)$  bounded by a geodesic  $\gamma$  of length  $2d$  and  $A$  a constant curvature  $\kappa = 2H$  arc joining the end points of  $\gamma$ . Then the area is*

$$A(L_H(2d)) = |L_H(2d)| = 2H|A| - 2 \arcsin(2H \tanh d).$$

*Proof.* By Gauss-Bonnet,  $|L_H(2d)| = 2H|A| - 2\alpha$  where  $\pi - \alpha$  is the exterior angle at the edge of the lens. We use the half-space model of the hyperbolic space. The geodesic segment of hyperbolic length  $2d$  is the vertical Euclidean segment between the points  $p_- = (0, 1 - \tanh d)$  and  $p_+ = (0, 1 + \tanh d)$  in  $R^{+3}$ . We consider an arc of a circle with radius  $R = 1/2H$ , passing by  $p_-$  and  $p_+$ . Then the center of the circle is  $O = (x, 1)$  with  $x^2 + \tanh^2 d = 1/4H^2$ . The hyperbolic curvature of the circle is  $\kappa = 2H$  (the point  $O$  is at height 1). The lens  $L_H(2d)$  is the region bounded by the arc and the vertical segment between  $p_-$  and  $p_+$ . Elementary trigonometry gives that  $\sin \alpha = 2H \tanh d$ .  $\square$

LEMMA 9.3. *We consider the function*

$$F(\alpha, \beta) = 2H(|T(\alpha)| + |L_H(2d_\alpha)| + |T(\beta)| - |L_H(2d_\beta)|) + |B| - |A|.$$

with  $(\alpha, \beta) \in [0, \pi]^2$  and  $\alpha + \beta = \theta \in [0, \pi]$ . Then we have  $F(0, \beta) > 0$  and for  $0 < \alpha < \gamma \leq \pi$ ,

$$0 > F(\alpha, 0) > F(\gamma, 0).$$

*Proof.* We consider  $F(\alpha, \beta) = 2H(|T(\alpha)| + |L_H(2d_\alpha)| + |T(\beta)| - |L_H(2d_\beta)|) + |B| - |A|$  for some value  $\alpha + \beta = \theta \in [0, \pi]$  and by the two lemmas above we have

$$\begin{aligned} F(\alpha, \beta) &= 4H \arcsin\left(\frac{\sinh c_\alpha \sinh d_\alpha}{1 + \cosh b}\right) + 4H \arcsin\left(\frac{\sinh c_\beta \sinh d_\beta}{1 + \cosh b}\right) \\ &\quad - 4H \arcsin(2H \tanh d_\alpha) + 4H \arcsin(2H \tanh d_\beta) \\ &\quad + (1 - 4H^2)(|B| - |A|). \end{aligned}$$

When  $\alpha = 0$ , we have  $d_\alpha = 0$  and then  $F(0, \beta) > 0$  when  $\pi > \beta > 0$ . When  $\beta = 0$ , we prove for  $\alpha \in (0, \pi)$

$$F(\alpha, 0) = 4H \arcsin \left( \frac{\sinh c_\alpha \sinh d_\alpha}{1 + \cosh b} \right) - 4H \arcsin (2H \tanh d_\alpha) + (4H^2 - 1)|A| < 0.$$

To see that we remark that  $F(0, 0) = 0$  by construction and we prove that  $F(\gamma, 0) < F(\alpha, 0)$  for  $0 < \alpha < \gamma < \pi$ .

We rewrite

$$\begin{aligned} F(\alpha, 0) - F(\gamma, 0) &= (1 - 4H^2)(|A_\gamma| - |A_\alpha|) \\ &\quad + 4H \arcsin \left( \frac{\sinh b}{1 + \cosh b} \tanh d_\alpha \cos \alpha/2 \right) \\ &\quad - 4H \arcsin \left( \frac{\sinh b}{1 + \cosh b} \tanh d_\alpha \cos \gamma/2 \right) \\ &\quad + 4H \arcsin \left( \frac{\sinh b}{1 + \cosh b} \tanh d_\alpha \cos \gamma/2 \right) \\ &\quad - 4H \arcsin \left( \frac{\sinh b}{1 + \cosh b} \tanh d_\gamma \cos \gamma/2 \right) \\ &\quad + 4H \arcsin (2H \tanh d_\gamma) \\ &\quad - 4H \arcsin (2H \tanh d_\alpha). \end{aligned}$$

When  $\alpha < \gamma$ , we have  $d_\alpha < d_\gamma$  and then  $|A_\alpha| < |A_\gamma|$ . Then the first line is positive. The second line of the equation above is positive ( $\cos \gamma/2 < \cos \alpha/2$ ). For the last two lines, we can see that the derivative of the function  $g(t) = 2 \arcsin (ta) - 2 \arcsin (tb)$  has the sign of  $a^2 - b^2$ . With  $a = \tanh d_\alpha$  and  $b = \tanh d_\gamma$ , we have

$$g \left( \frac{\sinh b \cos \gamma/2}{1 + \cosh b} \right) > g(1) > g(2H)$$

and  $F(\alpha, 0)$  is decreasing as desired. □

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## REFERENCES

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- [1] J. A. Aledo, J. H. Espinar, and J. A. Gálvez, Height estimates for surfaces with positive constant mean curvature in  $\mathbb{M}^2 \times \mathbb{R}$ , *Illinois J. Math.* (to appear).
  - [2] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, Berlin, 1983.
  - [3] H. Jenkins and J. Serrin, Variational problems of minimal surface type II: Boundary value problems for the minimal surface equation, *Arch. Rat. Mech. Anal.* **21** (1966), 321–342.
  - [4] B. Nelli and H. Rosenberg, Minimal surfaces in  $\mathbb{H}^2 \times \mathbb{R}$ , *Bull. Braz. Math. Soc.* **33** (2002), 263–292.
  - [5] J. C. C. Nitsche, Über ein verallgemeinertes Dirichletsches Problem für die Minimalflächengleichung und hebbare Unstetigkeiten ihrer Lösungen, *Math. Ann.* **158** (1965), 203–214.
  - [6] A. L. Pinheiro, The theorem of Jenkins-Serrin in  $\mathbb{M} \times \mathbb{R}$ , preprint, 2005.
  - [7] J. Serrin, The problem of Dirichlet for quasilinear elliptic differential equations with many independent variables, *Philos. Trans. Roy. Soc. London Ser. A* **264** (1969), 413–496.
  - [8] ———, The Dirichlet problem for surfaces of constant mean curvature, *Proc. London Math. Soc.* **21** (1970), 361–384.
  - [9] J. Spruck, Infinite boundary value problems for surfaces of constant mean curvature, *Arch. Rat. Mech. Anal.* **49** (1972), 1–31.
  - [10] ———, Interior gradient estimates and existence theorems for constant mean curvature graphs in  $\mathbb{M} \times \mathbb{R}$ , *Pure Appl. Math. Q.* **3** (2007), 785–800.
  - [11] S. Zhang, Curvature estimates for CMC surfaces in three dimensional manifolds, *Math. Z.* **249** (2005), 613–624.