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PROPERTIES OF THE SOLUTIONS OF THE CONJUGATE HEAT EQUATIONS

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Abstract. In this paper we consider the class $\mathcal{A}$ of those solutions $u(x,t)$ to the conjugate heat equation $\frac{\partial}{\partial t}u = -\Delta u + Ru$ on compact Kähler manifolds M with $c_1 > 0$ (where $g(t)$ changes by the unnormalized Kähler Ricci flow, blowing up at $T < \infty$), which satisfy Perelman's differential Harnack inequality (6) on $[0,T)$. We show $\mathcal{A}$ is nonempty. If $|\operatorname{Ric}(g(t))| \le \frac{C}{T-t}$, which is always true if we have a type I singularity, we prove the solution $u(x,t)$ satisfies the elliptic type Harnack inequality, with the constants that are uniform in time. If the flow $g(t)$ has a type I singularity at T, then $\mathcal{A}$ has exactly one element.

1. Introduction. Let M be a Kähler manifold with $c_1(M) > 0$, of complex dimension n. Consider the solutions to the unnormalized Kähler Ricci flow,

$$\frac{\partial}{\partial t} g_{i\bar{j}} = -R_{i\bar{j}}. \tag{1}$$

It is known in the case of the unnormalized Kähler Ricci flow that it shrinks to a point, after some finite time $T < \infty$. Let $T' < T$ and let

$$u = (4\pi(T'-t))^{-n} e^{-f},$$

satisfy the conjugate heat equation

$$\frac{\partial}{\partial t} u = -\Delta u + Ru. \tag{2}$$

This implies f satisfies,

$$\frac{\partial}{\partial t} f = -\Delta f + |\nabla f|^2 - R + \frac{n}{T'-t}. \tag{3}$$

Let

$$v = [(T'-t)(2\Delta f - |\nabla f|^2 + R) + f - 2n]u. \tag{4}$$

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It is easy to check that $\int_M v$ is exactly Perelman's functional $\mathcal{W}$. He proved it is monotonically increasing along the flow, that is,

$$\frac{d}{dt}\mathcal{W} = 2\tau \int_M |R_{i\bar{j}} + \nabla_i\nabla_{\bar{j}} f - g_{i\bar{j}}|^2 u\, dV \geq 0,$$

where $g(t)$ moves by the Kähler Ricci flow and $f(\cdot, t)$ evolves by (3). If u tends to a δ-function as $t \to T'$, in [8], Perelman proved $v \leq 0$ for all $t \in [0, T')$ (for a detailed proof of that see e.g. [12]). He also proved that under the same assumptions as above, for any smooth curve $\gamma(t)$ in M,

$$-\frac{\partial}{\partial t} f(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\dot{\gamma}(t)|^2) - \frac{1}{2(T' - t)} f(\gamma(t), t), \tag{5}$$

for all $t \in [0, T')$.

Definition 1. We say that a smooth solution f to

$$\frac{\partial}{\partial t} f = -\Delta f + |\nabla f|^2 - R + \frac{n}{T - t},$$

is *admissible* if

$$v(x, t) \leq 0, \ \text{ for } t \in [0, T), \tag{6}$$

where $v(x, t)$ is defined as in (4) with T' replaced by T. The metric $g(t)$ changes by the Ricci flow equation (1), having time T as the singular time at which the flow disappears. All the norms and the derivatives in (4) are defined with respect to the changing metric $g(t)$.

Notice that if $f(x, t)$ is admissible then it satisfies (5) with T' replaced by T.

Definition 2. Define

$$\mathcal{A} := \{u(x, t) = (4\pi(T - t))^{-n} e^{-f(x,t)} \ | \ u(x, t) \text{ solves (2) on } M \times [0, T) \text{ and } f \text{ is admissible}\}.$$

We will prove the following results about u.

THEOREM 3. *The set $\mathcal{A}$ is nonempty. If* $\mathrm{Ric}\,(g(t)) \geq -\frac{c}{T-t} g(t)$, *which translates to the condition,* $\mathrm{Ric} \geq -c$ *along the normalized Kähler Ricci flow, there is a uniform constant C, so that,*

$$\max_M u(x, t) \leq C \min_M u(x, t), \ \textit{for all } t \in [0, T).$$

If we assume the flow has a type I singularity, meaning that $|\,\mathrm{Rm}\,(g(t))| \le \frac{C}{T-t}$, *there is exactly one element in* $\mathcal{A}$, *that is, the solution to the conjugate heat equation (2), coming from T and satisfying (6) is unique.*

One of the corollaries of Theorem 3 is the following:

COROLLARY 4. *Let M be a compact Kähler manifold with* $c_1 > 0$ *and let* $g(t)$ *be a solution to the normalized Kähler Ricci flow* $\frac{\partial}{\partial t} g = g - \mathrm{Ric}\,(g)$. *If* $\mathrm{Ric}\,(g(t)) \ge -Cg(t)$ *uniformly along the flow, there exist a solution* $u(x,t)$ *to*

$$\frac{\partial}{\partial t} u = -\Delta u + (R-n)u, \text{ for all } t \in [0,\infty), \tag{7}$$

and a uniform constant $\tilde{C}$ *so that*

$$\max_M u(x,t) \le \tilde{C} \min_M u(x,t), \text{ for all } t \ge 0.$$

To motivate Theorem 3 and explain the possible use of the existence of $u \in \mathcal{A}$, consider $(M, g(t))$ given as in Corollary 4. Knowing the existence of $u \in \mathcal{A}$ we can give a shorter proof of the following result that can be found in [9].

COROLLARY 5. *Let* $(M, g(t))$ *be the normalized Kähler Ricci flow on a compact Kähler manfold with* $c_1 > 0$. *If* $|\,\mathrm{Rm}\,| \le C$ *for all* $t \ge 0$ *then for every sequence* $t_i \to \infty$ *there exists a subsequence so that* $(M, g(t_i + t))$ *converges as* $i \to \infty$ *to* $(M, h(t))$, *where* $h(t)$ *is a gradient Kähler Ricci soliton metric.*

Similarly as in [10], using our solution u, we get that if only $|\,\mathrm{Ric}\,|$ (instead of $|\,\mathrm{Rm}\,|$) is uniformly bounded along the flow, we have a sequential convergence of the normalized flow to gradient Kähler Ricci shrinking solitons away from a singular set S that is of real codimension at least 4. Our hope is that one may be able to use the properties of $u \in \mathcal{A}$ to obtain more information about the structure of the singular set S, which is still an interesting open problem in the Kähler Ricci flow theory. A function $u \in \mathcal{A}$ is a solution to the conjugate heat equation coming all the way from the singularity, bearing some information about singularity formation that can be used in better understanding its structure.

The organization of the paper is as follows. In section 2 we will give the proof of Theorem 3, Corollary 4 and Corollary 5. Complex two dimensional case will be discussed in section 3.

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2. Harnack type estimates and the uniqueness of u. We will assume for the moment that $\mathcal{A}$ is not empty and prove that each element $u(t) \in \mathcal{A}$ satisfies the elliptic Harnack inequality at each time slice with the uniform constant, not depending on time, and that such a solution is unique if $g(t)$ has a type I singularity at T.

PROPOSITION 6. *If* $\operatorname{Ric}(g(t)) \geq -\frac{C}{T-t}g(t)$ *along the flow* $g(t)$*, there exists a uniform constant* $\tilde{C}$*, so that*

$$\max_M u(x,t) \leq \tilde{C} \min_M u(x,t),$$

for all $t \in [0,T)$.

Proof. Take $t_1 < t_2 < T$ and $x_1, x_2 \in M$. Let $\gamma(t)$ be a curve that will be chosen later, so that it connects x_1 and x_2, that is, $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. Since f is an admissible function, it satisfies (6), for a chosen curve γ. Integrate it in $t \in [t_1, t_2]$. We get,

$$f(x_1,t_1)\sqrt{T-t_1} \leq f(x_2,t_2)\sqrt{T-t_2} + \frac{1}{2}\int_{t_1}^{t_2} \sqrt{T-t}(R(\gamma(t),t) + |\dot{\gamma}(t)|^2)\,dt. \tag{8}$$

By translation in time, we may assume $T = 1/2$. It easily follows that if we rescale the flow, that is, if $\tilde{g}(s(t)) = \frac{g(t)}{1-t/T} = Tg(t)/(T-t)$ with $s(t) = -T\ln(1-t/T)$, we get a normalized Kähler Ricci flow satisfying

$$\frac{\partial}{\partial s}\tilde{g} = \tilde{g} - \operatorname{Ric}(\tilde{g}),$$

for all $s \in [0,\infty)$. By Perelman's results (see [11] for complete proofs), $|R(\tilde{g}(s))| \leq C$ and $\operatorname{diam}(M, \tilde{g}(s)) \leq C$ along the flow. This implies

$$|R(g(t))| \leq \frac{C}{T-t}, \tag{9}$$

and

$$\operatorname{diam}(M, g(t)) \leq C\sqrt{T-t}. \tag{10}$$

Combining our condition $\operatorname{Ric}(g(t)) \geq -\frac{C}{T-t}g(t)$ together with (9) yields

$$|\operatorname{Ric}(g(t))| \leq \frac{C_1}{T-t},$$

for a uniform constant C_1 that is time independent. As a matter of reparametrization, we also get

$$\operatorname{Vol}_{g(t)} = (1-t/T)^n \operatorname{Vol}_{\tilde{g}(s(t))}(M) = C(T-t)^n. \tag{11}$$

We will estimate the integral term appearing in (8). By (9) we have,

$$\int_{t_1}^{t_2} \sqrt{T-t}R(\gamma(t),t)\,dt \leq C\int_{t_1}^{t_2} \frac{dt}{\sqrt{T-t}} = C\frac{t_2-t_1}{\sqrt{T-t_1}+\sqrt{T-t_2}}. \tag{12}$$

Without losing any generality assume that $|\operatorname{Ric}(g(t))|(T-t) \leq 1$, since the consideration otherwise would be similar. We have the simple claim that follows immediatelly from the evolution equation for $g(t)$.

Claim 7. If $\operatorname{Ric}(g(t))(T-t) \geq -g(t)$ for all $t \in [0,T)$, for any $0 \leq t < s < T$, we have,

$$g(s) \leq \frac{T-t}{T-s}g(t).$$

In particular, for any vector v, we have

$$|v|^2_{g(s)} \leq \frac{T-t}{T-s}|v|^2_{g(t)}.$$

Let $s_1 = s(t_1)$. If we choose γ to be a minimal geodesic from x_1 to x_2 with respect to $g(t_1)$, by (10) and by the previous claim, we have,

$$\begin{aligned} \int_{t_1}^{t_2} \sqrt{T-t}|\dot{\gamma}|^2\,dt &= \int_{T-t_2}^{T-t_1} \sqrt{\tau}|\dot{\gamma}|^2_{g(T-\tau)}\,d\tau \\ &\overset{\sqrt{\tau}=s}{=} \int_{\sqrt{T-t_2}}^{\sqrt{T-t_1}} \frac{1}{2}|\gamma'|^2_{g(T-s^2)}\,ds \\ &\leq C\int_{\sqrt{T-t_2}}^{\sqrt{T-t_1}} \frac{T-t_1}{T-t_2}|\gamma'|^2_{g(t_1)}\,ds \\ &= \tilde{C}\frac{T-t_1}{T-t_2}\frac{\operatorname{dist}^2_{g(t_1)}(x_1,x_2)}{\sqrt{T-t_1}-\sqrt{T-t_2}} \\ &\overset{(10)}{\leq} \tilde{C}\frac{(T-t_1)^2\sqrt{T-t_1}}{(T-t_2)(t_2-t_1)}. \end{aligned} \tag{13}$$

From estimates (8), (12) and (13), we get

$$f(x_1,t_1) \leq \frac{\sqrt{T-t_2}}{\sqrt{T-t_1}}f(x_2,t_2) + C\frac{t_2-t_1}{T-t_1} + C\frac{(T-t_1)^2}{(t_2-t_1)(T-t_2)}.$$

If $(*)$ t_1,t_2 are such that $\lambda(T-t_1) \leq t_2-t_1 \leq \mu(T-t_1)$, for some uniform constants λ,μ, e.g. $t_2-t_1 = T-t_2 = \delta$, then,

$$f(x_1,t_1) \leq \frac{1}{\sqrt{2}}f(x_2,t_2) + C, \tag{14}$$

and since x_1, x_2 were two arbitrary points, we have

$$\max_M f(\cdot, t_1) \leq \frac{1}{\sqrt{2}} \min_M f(\cdot, t_2) + C. \tag{15}$$

In general, the constant C in (14) is independent of time and depends on our choice of λ, μ.

We claim there is some $\tilde{A}$ so that $f(x,t) \geq -\tilde{A}$, for all $x \in M$ and all $t \in [0,T)$. Assume $\max_M f(\cdot,t) \leq -A$. That implies $f(x,t) \leq -A$, for all $x \in M$ and therefore,

$$(4\pi\tau(T-t))^n = \int_M e^{-f}\, dV_t \geq e^A \operatorname{Vol}_t(M) = e^A C(T-t)^n,$$

for a uniform constant C, which is not possible for big enough A (notice that the bigness of A does not depend on $t \in [0,T)$). Estimate (15) now implies

$$\min_M f(\cdot, t_2) \geq -\tilde{A},$$

for a uniform constant $\tilde{A}$, independent of t_2. Rewrite (14) as

$$\begin{aligned} \max_M f(\cdot, t_1) &\leq \frac{1}{\sqrt{2}}(\min_M f(\cdot, t_2) + \tilde{A}) - \frac{\tilde{A}}{\sqrt{2}} + C \\ &\leq \min_M f(\cdot, t_2) + \tilde{A} + C_1 = \min_M f(\cdot, t_2) + C_2. \end{aligned}$$

If we denote by $M(t) = \max_M u(\cdot,t)$ and by $m(t) = \min_M u(\cdot,t)$, this yields

$$M(t_2) \leq C(\frac{T-t_1}{T-t_2})^n m(t_1) \leq \tilde{C} m(t_1), \tag{16}$$

where $\tilde{C}$ in general depends on a choice of λ, μ in $(*)$. By the maximum principle applied to $\frac{\partial}{\partial t}u = -\Delta u + Ru$, at the maximum point of $u(\cdot,t)$,

$$\frac{d}{dt}M(t) \geq RM(t) \geq -\frac{C}{T-t}M(t),$$

which yields $M(t_2) \geq (\frac{T-t_2}{T-t_1})^C M(t_1) = \tilde{C}M(t_1)$. This together with (16) gives,

$$M(t_1) \leq Cm(t_1), \tag{17}$$

for a uniform constant C and all $t_1 \in [0,T)$, which is an analogue of the Harnack inequality that we have in the elliptic case. ∎

We will now prove the nonemptiness of $\mathcal{A}$ without assuming anything on the curvature.

LEMMA 8. *For every Kähler manifold M and every unnormalized Kähler Ricci flow $g(t)$, $|\mathcal{A}| \geq 1$.*

Proof. For every unnormalized Kähler Ricci flow $g(t)$ there is a finite time T at which the flow disappears. Take an arbitrary increasing sequence of times $t_i \uparrow T$ and a sequence of points $p_i \in M$. For every i, let $u_i(t) = (4\pi(t_i - t))^{-n} e^{-f_i(t)}$ be a solution to the conjugate heat equation (2), such that $u_i(t)$ converges to a δ-function as $t \to t_i$, concentrated at p_i. Let $v_i(t)$ be a corresponding v-function as in (4). Due to Perelman (see [8]), we have $v_i(t) \leq 0$ for all $t \in [0, t_i]$ and for every smooth curve $\gamma(t)$,

$$-\frac{\partial}{\partial t} f_i(\gamma(t), t) \leq \frac{1}{2}(R(\gamma(t), t) + |\dot{\gamma}(t)|^2) - \frac{1}{2(t_i - t)} f_i(\gamma(t), t),$$

holds for all $t \in [0, t_i]$, which as before, implies

$$f_i(x, t_1)\sqrt{t_i - t_1} \leq f_i(y, t_2)\sqrt{t_i - t_2} + \frac{1}{2}\int_{t_1}^{t_2} \sqrt{t_i - t}(R(\gamma(t), t) + |\dot{\gamma}|^2)\, dt, \tag{18}$$

for $x, y \in M$ and $t_1 \leq t_2 < t_i$.

Fix t_j from the sequence of times and consider $u_i(t)$ for those $i \geq i_j$ for which $t_i \geq \frac{T+t_j}{2}$. Our goal is to show that we have uniform estimates on $u_i(\cdot, t)$ and its derivatives for $t \in [0, t_j]$, that do not depend on $i \geq i_j$. This will enable us extract a convergent subsequence of our solutions $u_i(\cdot, t)$ for $t \in [0, t_j]$. To obtain those estimates, notice first that the metrics $g(t)$ have the property that $|\operatorname{Ric}(g(t))| \leq C_j$ on $M \times [0, t_j]$ and therefore are uniformly equivalent. Take any $x, y \in M$ and $t < t_j$. Then (18) implies

$$f_i(x, t) \leq f_i(y, t_j) + C_j, \text{ for all } i \geq i_j \text{ and all } t \in [0, t_j], \tag{19}$$

using the similar analysis as in the proof of Proposition 6 and the fact that $|\operatorname{Ric}(g(t))| \leq C_j$ for $t \in [0, t_j]$ (this Ricci curvature bound implies that for any vector v we have $|v|^2_{g(s)} \leq \bar{C}_j |v|^2_{g(t)}$ for $t < s \leq t_j$, which we use in obtaining (19)). Since $(4\pi(t_i - t_j))^{-n} \int_M e^{-f_i(x, t_j)}\, dV_{t_j} = 1$, we can find a uniform constant $\bar{C}_j$ and at least 2 points, x_{ij}, y_{ij}, so that $f_i(x_{ij}, t_j) \leq \bar{C}_j$ and $f_i(y_{ij}, t_j) \geq -\bar{C}_j$, for all $i \geq i_j$. Take any $x \in M$ and x_{ij} from before. Estimate (19) yields

$$f_i(x, t) \leq f_i(x_{ij}, t_j) + C_j \leq \bar{C}_j + C_j =: \tilde{C}_j, \text{ for all } (x, t) \in M \times [0, t_j] \text{ and } i \geq i_j. \tag{20}$$

In a similar manner, using (19) we get

$$f_i(y,t_j) \geq -\tilde{C}_j, \text{ for all } y \in M, \text{ and } i \geq i_j. \tag{21}$$

Moreover,

$$\frac{d}{dt}f_i = -\Delta f_i + |\nabla f_i|^2 - R + \frac{n}{t_i - t},$$

which implies,

$$\frac{d}{dt}(f_i)_{\min} \leq \frac{C}{T-t} + \frac{n}{t_i - t} \leq \frac{\tilde{C}}{t_i - t},$$

and therefore, by (21),

$$(f_i)_{\min}(t_j) \leq (f_i)_{\min}(t) + \tilde{C}\ln\left(\frac{t_i - t}{t_i - t_j}\right), \text{ for } t \in [0,t_j].$$

This yields,

$$(f_i)_{\min}(t) \geq -C(t_j), \tag{22}$$

for $i \geq i_j$ and henceforth, (20) and (22) imply

$$0 < \delta_j \leq u_i(x,t) \leq \hat{C}_j, \text{ on } M \times [0,t_j], \text{ for } i \geq i_j, \tag{23}$$

for some uniform constants $\delta_j, \hat{C}_j$. If we denote by $\tilde{u}_i(x,t) = u_i(x,t)^{1/2}$, integrating $v_i(x,t) \leq 0$, similarly as in [12], by using Hölder and Sobolev inequalities we get

$$\int_M |\nabla \tilde{u}_i|^2 \, dV_t \leq C_j \text{ for } t \in [0,t_j] \text{ and } i \geq i_j,$$

which together with (23) imply,

$$\sup_{t\in[0,t_j]} \|u_i(t)\|_{W^{1,2}} \leq C_j,$$

for all i big enough. By standard parabolic estimates applied to $u_i(t)$ and $t \in [0,t_j]$, it follows there exists $C(k,l,n,t_j)$ so that

$$\left|\frac{\partial^l}{\partial t^l}u_i(t)\right|_{C^k} \leq C(k,l,n,t_j),$$

for all $t \in [0,t_j]$ and all i sufficiently big. Extract a subsequence $u_i(t,x)$ that converges in $C^{k,l}(M \times [0,t_j])$ norm as $i \to \infty$ to some function $u(t,x) \geq \delta_j > 0$, defined on $[0, t_j]$ that continues to be a solution to the conjugate heat equation (2), where the lower bound on u follows from (23). By taking larger and larger j, diagonalizing our sequence $u_i(t)$, taking into account the uniqueness of the limit, we get a function $u(t,x) > 0$, defined on $M \times [0,T)$ and a subsequence $u_i(t,x)$, so that $u_i(t,x) \overset{C^{k,l}(M\times[0,T'])}{\rightarrow} u(t,x)$, for every $T' < T$. Moreover,

- $u(t,x)$ satisfies the conjugate heat equation (2) for all $t \in [0,T)$.
- $u(t,x) = (4\pi(T-t))^{-n}e^{-f}$, where f is an admissable function in the sense of Definition 1.

In particular, this implies $|\mathcal{A}| \geq 1$ □

We will now prove the uniqueness part of Theorem 3.

PROPOSITION 9. *If $g(t)$ has a type I singularity at T then $|\mathcal{A}| = 1$, that is, the solution u to the conjugate heat equation $\frac{\partial}{\partial t}u = -\Delta u + Ru$, that satisfies Perelman's differential Harnack inequality (6) is unique.*

Proof. Assume there are at least two different solutions u_1 and u_2, with the properties as above. By Proposition 6, we have $M_j(t) \leq Cm_j(t)$, for all $t \in [0,T)$ and $j \in \{1,2\}$. We will omit the subscript j below. On the other hand, we have the integral normalization condition $\int_M u\, dV_{g(t)} = 1$. Combining these two facts, we get

$$m(t) \geq \frac{1}{C\,\mathrm{Vol}_{g(t)}(M)}, \tag{24}$$

$$M(t) \leq C\frac{1}{\mathrm{Vol}_{g(t)}(M)}, \tag{25}$$

where a constant C comes from (17). Take a sequence $t_i \to T$ and consider a sequence of rescaled metrics $g_i(t) = (T-t_i)^{-1}g((T-t_i)t + t_i)$, for $t \in [-(T-t_i)^{-1}t_i, 1)$. Since $|\mathrm{Rm}(g(t))| \leq \frac{C}{T-t}$ (due to the fact that $g(t)$ has the type I singularity at T) we get

$$\begin{aligned} |\mathrm{Rm}(g_i(t))| &= |\mathrm{Rm}(g(t(T-t_i)+t_i)|(T-t_i) \\ &\leq C\frac{(T-t_i)}{T-t_i-t(T-t_i)} \\ &\leq C\frac{1}{1-t} \leq \tilde{C}, \end{aligned} \tag{26}$$

for all $t \in [-(T-t_i)^{-1}t_i, 1/2]$. By Perelman's volume noncollapsing result and by Hamilton's compactness theorem, there is a subsequence $(M, g_i(t))$, converging to an ancient Kähler Ricci solution $(M, h(t))$, defined for $t \in (-\infty, 1/2]$. If we also rescale our solution u, together with our metric $g(t)$, estimates (24), (25) and

(11) give

$$\begin{aligned} u_i(t) &= (T-t_i)^n u(t(T-t_i)+t_i) \\ &\le C\frac{(T-t_i)^n}{\mathrm{Vol}_{g(t(T-t_i)+t_i)}(M)} \\ &= \tilde{C}\frac{(T-t_i)^n}{(T-t_i)^n(1-t)^n} \le \bar{C}, \end{aligned}$$

for all $t \in [-(T-t_i)^{-1}t_i, 1/2]$. Notice that function f rescales as $f_i(t) = f(t_i + t(T-t_i))$. Similarly we get the uniform lower bound on $u_i(t)$, that is, there exists a uniform constant $\tilde{C} > 1$ so that

$$\frac{1}{\tilde{C}} \le u_i(t) \le \tilde{C}, \tag{27}$$

on $M \times [-(T-t_i)^{-1}t_i, 1/2]$.

Since $f \in \mathcal{A}$ we have

$$((T-t)(2\Delta f - |\nabla f|^2 + R) + f - n)u \le 0, \text{ on } M \times [0, T).$$

As a matter of scaling we get

$$[(1-t)(2\Delta_i f_i - |\nabla_i f_i|^2 + R(g_i)) + f_i - n]u_i \le 0.$$

If we integrate the previous inequality over M by parts, use (27) and (26) we get

$$\sup_{t\in[-(T-t_i)^{-1}t_i,1/2]} \int_M |\nabla u_i(x,t)|^2 \, dV_{g_i(t)} \le \tilde{C},$$

for a uniform constant $\tilde{C}$, independent of i.

Functions $u_i(t)$ satisfy backward parabolic equations

$$\frac{\partial}{\partial t}u_i(t) = -\Delta u_i(t) + R(g_i(t))u_i(t).$$

By standard parabolic estimates applied to u_i, we have

$$\sup_{t\in[-1,1/4]} |u_i|_{C^k(M)} \le C(k),$$

uniformly in i. Combining this together with (27), we have

$$\sup_{t\in[-1,1/4]} |f_i(t)|_{C^k} \le C(k). \tag{28}$$

Claim 10. Under the same assumptions as in Proposition 9 there is a uniform constant C so that

$$\sup_{t\in[0,T)} \mathcal{W}(g(t),f(t),T-t) \leq C.$$

Proof. Consider our sequence $t_i \to T$. Then, since $\mathcal{W}$ is invariant under scaling, by estimates (28),

$$\begin{aligned} \mathcal{W}(g(t_i),f(t_i),T-t_i) &= \mathcal{W}(g_i(0),f_i(0),1) \\ &= (4\pi)^{-n}\int_M (2(R_i+|\nabla f_i|^2)+f_i-2n)e^{-f_i}\,dV_{g_i(0)} \\ &\leq C. \end{aligned} \tag{29}$$

By Perelman's monotonicity formula $\mathcal{W}(g(t),f(t),T-t)$ increases in time, which together with (29) imply

$$\mathcal{W}(g(t),f(t),T-t) \leq \mathcal{W}(g(t_i),f(t_i),T-t_i) \leq C, \text{ for all } t\in[0,t_i]. \tag{30}$$

Since (30) holds for every i we have the statement of the claim. □

The previous claim and Perelman's monotonicity formula for $\mathcal{W}$ yield the existence of a finite limit, $\lim_{t\to T}\mathcal{W}(g(t),f(t),T-t)$. From before, we have that $g_i(t)\to h(t)$. From our estimates (28) on $f_i(s)$, by extracting a subsequence we may assume $f_i(s) \overset{C^k(M\times[-1,1/4]}{\longrightarrow} f_h(s)$. Let $a_i = \frac{T+t_i}{2} > t_i$. Then

$$\begin{aligned} &\mathcal{W}(g(a_i),f(a_i),T-a_i) - \mathcal{W}(g(t_i),f(t_i),T-t_i) \\ &= \int_{t_i}^{a_i}\frac{d}{dt}\mathcal{W}\,dt \\ &= \int_{t_i}^{a_i}(4\pi(T-t))^{-n}\int_M (2(T-t)|R_{p\bar q}+\nabla_p\nabla_{\bar q}f-g_{p\bar q}|^2e^{-f}\,dV_{g(t)}\,dt \\ &\quad +|\nabla_p\nabla_q f|^2+|\nabla_{\bar p}\nabla_{\bar q}f|^2)e^{-f}dV_{g(t)}dt \\ &\geq (4\pi(T-t_i))^{-n}\int_{t_i}^{a_i}\int_M((T-t_i)(|R_{p\bar q}+\nabla_p\nabla_{\bar q}f-g_{p\bar q}|^2+|\nabla_p\nabla_q f|^2 \\ &\quad +|\nabla_{\bar p}\nabla_{\bar q}f|^2)e^{-f}\,dV_{g(t)}\,dt \\ &= (4\pi)^{-n}\int_0^{1/2}\int_M(|\operatorname{Ric}_i+\nabla\bar\nabla f_i-g_i|^2+|\nabla\nabla f_i|^2+|\bar\nabla\bar\nabla f_i|^2)e^{-f_i}\,dV_{g_i(s)}\,ds \\ &\geq (4\pi)^{-n}\int_0^{1/4}\int_M(|\operatorname{Ric}_i+\nabla\bar\nabla f_i-g_i|^2+|\nabla\nabla f_i|^2+|\bar\nabla\bar\nabla f_i|^2)e^{-f_i}\,dV_{g_i(s)}\,ds. \end{aligned} \tag{31}$$

The left-hand side of (31) converges to zero, while its right-hand side converges to

$$\int_0^{1/4}\int_M (|\operatorname{Ric}(h) + \nabla\bar{\nabla} f_h - h|^2 + |\nabla\nabla f_h|^2 + |\bar{\nabla}\bar{\nabla} f_h|^2) e^{-f_h}\, dV_{h(s)}\, ds.$$

This yields $h(s)$ is a Kähler Ricci soliton and it satisfies,

$$\begin{aligned} R_{p\bar{q}}(h) + \nabla_p \nabla_{\bar{q}} f_h - h_{p\bar{q}} &= 0,\\ f_{\bar{p}\bar{q}} = f_{pq} &= 0. \end{aligned}$$

In other words, what we get is the following: if we have two different solutions $u_1 = (4\pi(T-t))^{-n} e^{-f_1}$ and $u_2 = (4\pi(T-t))^{-n} e^{-f_2}$, to each of them we can apply the reasoning from above. We can consider $u_i^1(t) = (T-t_i)^{-n} u_1(t_i + t(T-t_i))$ and $u_i^2(t) = (T-t_i)^{-n} u_2(t_i + t(T-t_i))$ and as above, we conclude $f_1(t_i + t(T-t_i)) \stackrel{C^k}{\to} f_h^1$ and $f_2(t_i + t(T-t_i)) \stackrel{C^k}{\to} f_h^2$, where f_h^1 and f_h^2 both satisfy,

$$\operatorname{Ric}(h) + \nabla\bar{\nabla} f_h^1 - h = 0,$$

$$\operatorname{Ric}(h) + \nabla\bar{\nabla} f_h^2 - h = 0.$$

This implies $\Delta f_h^1 = \Delta f_h^2$, which yields $f_h^1 = f_h^2 + C$, for some constant C. Since $\int_M e^{-f_h^1}\, dV_h = \int_M e^{-f_h^2}\, dV_h$, we get $C = 0$. This means, for $t \in [-1, 1/2]$,

$$\frac{u_1(t_i + t(T-t_i))}{u_2(t_i + t(T-t_i))} = \frac{u_i^1(t)}{u_i^2(t)} \stackrel{C^k}{\to} 1,$$

and in particular, by putting $t = 0$,

$$\frac{u_1(t_i)}{u_2(t_i)} = \frac{u_i^1(0)}{u_i^2(0)} \stackrel{C^k}{\to} 1, \tag{32}$$

as $i \to \infty$, where $u_i^1(t) = (T-t_i)^n u_1(t_i + t(T-t_i))$, and similarly for $u_i^2(t)$.

A simple computation shows that the evolution equation for $\frac{u_1}{u_2}$, since both functions u_1 and u_2 satisfy the conjugate heat equation (2) is

$$\frac{\partial}{\partial t}\frac{u_1}{u_2} = -\Delta\frac{u_1}{u_2} - \nabla \ln u_2 \nabla\left(\frac{u_1}{u_2}\right). \tag{33}$$

If there is a time $t_0 \in [0, T)$ and $x \in M$ so that $\frac{u_1(x,t_0)}{u_2(x,t_0)} \geq 1 + \delta$, with $\delta > 0$, then $\max_M (\frac{u_1(\cdot,t_0)}{u_2(\cdot,t_0)}) \geq 1 + \delta$. By the maximum principle applied to (33), we get

$\max_M (\frac{u_1(\cdot,t)}{u_2(\cdot,t)})$ increases in time and therefore,

$$\max_M \left(\frac{u_1(\cdot,t)}{u_2(\cdot,t)}\right) \geq 1+\delta,$$

for all $t \in [t_0, T)$. This contradicts (32). Similarly if there is a t_0 so that $\min_M \frac{u_1(\cdot,t_0)}{u_2(\cdot,t_0)} < 1-\delta < 1$, by the maximum principle applied to (33) we get $\min_M \frac{u_1(\cdot,t)}{u_2(\cdot,t)}$ decreases in time and therefore

$$\min_M \frac{u_1(\cdot,t)}{u_2(\cdot,t)} < 1-\delta \text{ for all } t \in [t_0, T),$$

which again contradicts (32). Henceforth $u_1(\cdot,t) = u_2(\cdot,t)$ for all $t \in [0,T)$. □

Proof of Theorem 3. The proof of the theorem follows from Lemma 8, Proposition 6 and Proposition 9. □

Corollary 4 is an immediate consequence of Theorem 3, it follows from the Theorem simply by rescalling and normalizing the volume of M along the flow.

Proof of Corollary 5. Take a solution $u = (2\pi)^{-n}e^{-f}$ to

$$\frac{\partial}{\partial t}u = -\Delta u + (R-n)u, \text{ for } t \in [0,\infty),$$

whose existence has been established by Corollary 4. It has the properties that

(i) $\max_M u(\cdot,t) \leq C \min_M u(\cdot,t)$, for all $t \geq 0$, and

(ii) $\int_M u\, dV_t = (2\pi)^n$, for all $t \geq 0$.

Since $\mathrm{Vol}_t(M) = const$, (i) and (ii) imply there are uniform constants c, C so that

$$c \leq u(x,t) \leq C, \text{ on } M \times [0,\infty).$$

Let $\tilde{g}(\cdot,s)$ be the unnormalized Kähler Ricci flow with the same initial data $\tilde{g}(\cdot,0) = g(\cdot,0)$, which exists up to an extinction time $T < \infty$. Then $\tilde{g}(\cdot,s) = (1-\frac{s}{T})g(t(s))$ where $t(s) = -T\ln(1-\frac{s}{T})$. Since $\mathcal{W}$ is invariant under scaling, by Claim 10 if we take the scaling factor to be $(T-s)/T$, we get

$$\mathcal{W}(g(t), f(t), T) \leq C, \text{ for all } t \geq 0. \tag{34}$$

By Perelman's results $\mathcal{W}(g(t), f(t), T)$ is increasing in t and therefore by (34) there is a finite $\lim_{t\to\infty} \mathcal{W}(g(t), f(t), T)$. Take an arbitrary sequence $t_i \to \infty$ and consider a sequence of solutions $g_i(t) = g(t_i + t)$. Fix a time interval $t \in [0, A]$ where A can be arbitrarily big. By Hamilton's compactness theorem (since the curvatures are uniformly bounded by our assumption and the injectivity radii are uniformly bounded from below by Perelman's volume noncollpasing theorem for the normalized Kähler Ricci flow, see [11]) there exists a subsequence so that $(M, g(t_i + t)) \to (M, h(t))$, where $h(t)$ solves the normalized Kähler Ricci flow equation. As in the proof of Proposition 9 we get uniform estimates on $f(\cdot, t_i + t)$ and their derivatives for all $t \in [0, A]$, and therefore by the Arzela Ascoli theorem we can find a subsequence t_i (we use the same notation for the sequence and the subsequence) so that

$$f(\cdot, t_i + t) \overset{C^k(M\times[0,A])}{\to} f_h(\cdot, t) \text{ as } i \to \infty. \tag{35}$$

$$\begin{aligned} (36) \qquad &= \mathcal{W}(g(t_i + A), f(t_i + A), T) - \mathcal{W}(g(t_i), f(t_i), T) \\ &= \int_0^A \frac{d}{ds}\mathcal{W}(g(t_i + s), f(t_i + s), T)\, ds \\ &= (4\pi T)^{-n} \cdot 2T \int_0^A \int_M (|R_{j\bar{k}}(t_i + s) + \nabla_j \bar{\nabla}_k f(t_i + s) - g_{j\bar{k}}(t_i + s)|^2 \\ &\quad + |\nabla_j \nabla_{\bar{k}} f(t_i + s)|^2 + |\bar{\nabla}_j \bar{\nabla}_k f(t_i + s)|^2) e^{-f(t_i+s)}\, dV_{t_i+s}\, ds. \end{aligned}$$

Since the left-hand side in (36) tends to zero as $i \to \infty$, since $g(t_i + t) \to h(t)$ in C^k-norm and since we have (35), we get

$$\mathrm{Ric}\,(h(\cdot, t)) + \nabla\bar{\nabla} f_h(\cdot, t) - h(\cdot, t) = 0 \text{ and } \nabla\nabla f_h = \bar{\nabla}\bar{\nabla} f = 0.$$

This finishes the proof of Corollary 5. □

3. More on the uniqueness of u for $n = 2$. In this section we will consider two dimensional Kähler, compact manifolds M, with $c_1 > 0$. Let $g(t)$ be the Kähler Ricci flow on such manifold. In the Kähler case, the curvature integral $\int_M |\mathrm{Rm}|^2\, dV$ is always bounded in terms of topological invariants, the first and the second Chern class. This integral is scale invariant for $n = 2$ which implies its significant importance in that case. For example, using that in [10] the following result has been proved.

THEOREM 11. *Let $g(t)$ be the normalized Kähler Ricci flow on a manifold as above, with uniformly bounded Ricci curvatures along the flow. Then for every sequence $t_i \to \infty$, there is a subsequence, so that $(M, g(t_i + t)) \to (M_\infty, g_\infty(t))$, where M_∞ is the orbifold with finitely many isolated singularities and $g_\infty(t)$ is a singular metric that satisfies the Kähler Ricci soliton equation outside those singular points.*

Combining Theorem 11 and Theorem 3 in the case of complex surfaces yields the uniqueness of the solution to the conjugate heat equation coming all the way from a singularity, under somewhat weaker assumption than having a type I singularity. In other words we have the following result:

THEOREM 12. *If $g(t)$ is the unnormalized Kähler Ricci flow on a manifold M as above, such that* $\mathrm{Ric}\,(g(t)) \geq -\frac{C}{T-t}g(t)$*, for a uniform constant C, there is a unique solution of the conjugate heat equation (2).*

Proof. The proof is analogous to the proof of Theorem 3, since the only singularities we get in two dimensional case are just isolated points. Adopt the notation from the proof of Theorem 3. First notice that our assumption on the lower bound on $\mathrm{Ric}\,(g(t))$ together with Perelman's bound on the scalar curvature imply $|\,\mathrm{Ric}\,(g(t))| \leq \frac{C}{T-t}$. For the rescaled sequence of metrics $g_i(t) = (T-t_i)^{-1}g(t_i + t(T-t_i))$, we have that

$$\sup_{M\times[-t_i(T-t_i)^{-1},1)} |\,\mathrm{Ric}\,(g_i(t))| \leq C,$$

$$\delta < u_i(t) \leq C,$$

for all $t \in [-t_i(T-t_i)^{-1}, \frac{1}{2}]$ and therefore,

$$\int_{-1}^{1/2}\int_M |\nabla f_i(s)|^2\, dV_{g_i(s)}\, ds \leq C.$$

The last estimate implies that for every i, there is $s_i \in [-1, 1/2]$, so that

$$\int_M |\nabla f_i(s_i)|^2\, dV_{g_i(s_i)} \leq C.$$

In the proof of Claim 10, to prove the boundness of $\mathcal{W}$, instead of considering $\mathcal{W}(g(t_i), f(t_i), T-t_i)$ we will consider

$$\mathcal{W}(g(t_i + s_i(T-t_i)), f(t_i + s_i(T-t_i)), (T-t_i)(1-s_i)), \text{ for } s_i \in [-1, 1/2].$$

The rest of the proof is same. We also have the same estimate (31) as before, where the left-hand side tends to zero as $i \to \infty$, due to the monotonicity and the boundness of $\mathcal{W}$. Assume $p_1, \dots, p_N$ are the singular points we get by taking the limit of the sequence $(M, g(t_i + t))$, and that $p_1^i, \dots, p_N^i$ are the curvature concentration points that are responsible for singularity formation in the limit (see [10]). Let $\{D_j\}$ be the compact exhaustion of $M_\infty \backslash \{p_1, \dots, p_N\}$. Our geometries $g(t)$ are uniformly bounded on each of D_j, (those bounds deteriorate when $j \to \infty$, that is, when we approach singularities). Henceforth, the estimate (31) tells us we can extract a subsequence, such that $(M, g(t_i + t)) \to (M_\infty, h(t))$ and $h(t)$ satisfies the Kähler Ricci soliton equation,

$$\mathrm{Ric}\,(h(t)) + \nabla\bar{\nabla} f_h(t) - h(t) = 0, \tag{37}$$

away from singular points. As in Proposition 9, if we assume there are at least two different solutions of the conjugate heat equation, we will get at least two different functions $f_h(t)$ and $f'_h(t)$, that satisfy (37) away from singular points. Without loss of generality assume there is only one singular point p.

Claim 13. Functions $f_h(t)$ and $f'_h(t)$ from above coincide on $M_\infty \backslash \{p\}$

Proof. Choose a sequence $\eta_k \to 1$ on M_∞ with $\int_{M_\infty} |\nabla \eta_k|_h^2 \, dV_h \to 0$ as $k \to \infty$, e.g.,

$$\eta_k(t) = \begin{cases} 0, & \text{for } x \in B(p, 1/k^2), \\ 1 - \frac{\ln(k^2 \operatorname{dist}_h(p,x))}{\ln k}, & \text{for } x \in B(p,1/k) \backslash B(p, 1/k^2), \\ 1, & \text{for } x \in M_\infty \backslash B(p, 1/k). \end{cases}$$

Denote by $F = f_h - f'_h$. It satisfies, $\Delta_h F = 0$ away from p. Multiply it by $F^2 \eta_k^2$ and then integrate over M. We get,

$$\begin{aligned} \int |\nabla F|^2 \eta_k^2 \, dV_h &= -\int \nabla \eta_k \eta_k F \nabla F \, dV_h \\ &\le 4 \int |F|^2 |\nabla \eta_k|^2 \, dV_h + \frac{1}{4} \int \eta_k^2 |\nabla F|^2 \, dV_h, \\ &\le C \int |\nabla \eta_k|^2 \, dV_h + \frac{1}{4} \int \eta_k^2 |\nabla F|^2 \, dV_h, \end{aligned}$$

since $\sup_{M_\infty \backslash \{p\}} |F| \le C$ (due to our estimates on f_i). After taking $k \to \infty$ we get,

$$\int_{M_\infty \backslash \{p\}} |\nabla F|^2 \, dV_h = 0.$$

As in [1], [2] and [13] one can show $M_\infty \backslash \{p\}$ is connected. This amounts to having $F = \textit{const}$ on $M_\infty \backslash \{p\}$. Becuase of the integral normalization condition for f_h and f'_h, we have $C = 0$ and therefore, $f_h = f'_h$. □

The rest of the proof of Theorem 12 is the same as that of Theorem 3. □

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