Paucity, abundance, and the theory of number: Online Appendices

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PAUCITY, ABUNDANCE, AND THE THEORY OF NUMBER: ONLINE APPENDICES

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APPENDIX A: INTRODUCTION, TERMINOLOGY, NOTATION

The following appendices present some formal considerations that were passed over in the main text so as not to impede the flow of the argument or upset readers of delicate constitution. The results presented are not to be read as mathematics that the child must perform. All the child does is place a bound somewhere in its lattices. The mathematics is what the linguist uses to verify that the properties of such cuts are what I have claimed them to be and that there are no lurking pathologies within the system.

The term *lattice* is consistently used to refer to complete power sets (not, as at times above, to abbreviate 'atomic complete join-semilattice').

Propositional versus presuppositional content will be represented in a fraction notation, in which the asserted content is the numerator and the presupposed content is the denominator (Harbour 2011a). For instance:

(A1) $[\pm\text{additive}] = \lambda \mathrm{P}\ \lambda x\ (\neg)\dfrac{\forall y(\mathrm{Q}(y) \rightarrow \mathrm{Q}(x \sqcup y))}{\mathrm{Q}(x) \wedge \mathrm{Q} \sqsubset \mathrm{P}}$ ← *proposition* ← *presupposition*

(A2) $[\pm\text{minimal}] = \lambda \mathrm{P}\ \lambda x\ (\neg)\neg\exists y \dfrac{\mathrm{P}(y) \wedge x \sqsubset y}{\mathrm{P}(x)}$ ← *proposition* ← *presupposition*

The two advantages of this notation are that it provides clear, and literal, delineation of assertion from presupposition (cf. Sauerland 2005) and straightforward arithmetic interpretability. As a brief example of the latter, observe that, arithmetically, $\frac{n}{1} = n$ but $\frac{n}{0}$ is undefined. The same is true if these expressions are read semantically: if the denominator equals 1 (if the presupposition is true), then the whole expression has just the value of the numerator (i.e. is truth-functionally equivalent to the proposition), and, if the denominator equals 0 (if the presupposition is false), then the whole expression is undefined. See Harbour 2011a for more complex examples, including the trial.

APPENDIX B: PERMUTATION INVARIANCE AND HORIZONTAL CUTS

STATEMENT 1. *A lattice region* Q *is permutation-invariant only if it is defined by horizontal cuts.*

PROOF. If nonhorizontal cuts are permitted, then there exist a and b of identical cardinality such that a is in Q and b is not. Clearly, then, we can permute the atoms of the lattice so as to 'swap the positions' of a and b, bringing b into Q at the expense of a. (Minimally, we just map $a\backslash(a \sqcap b)$ and $b\backslash(a \sqcap b)$ onto each other. Any such mapping is onto because $|a| = |b|$ entails that $|a\backslash(a \sqcap b)| = |b\backslash(a \sqcap b)|$.) ■

EXAMPLE. Label the three rightmost atoms in Fig. 6a x, y, and z. So, $a = x \sqcup y$ is Q and $b = y \sqcup z$ is not. Then, if we swap x and z around, without altering the shape of the cut that produces Q, a will now be outside Q and b in it.

When feature recursion was introduced (§3.3), it was claimed that (+additive(−additive(P))) was unsatisfiable because it 'looked for' the join-complete proper subregion within a join-incomplete region of

P. In light of statement 1, we can clarify this. The issue is not that a join-incomplete region cannot contain a join-complete one. For example, in the previous paragraph, Q contains the join-complete subregion $\{x, y, xy\}$. However, this could not be the [+additive] subregion of Q sinc it is not a horizontal cut: it contains some but not all atoms and dyads. To state the result more rigorously, we require the following concept:

> *Definition. A lattice region,* P, *is* PSEUDOATOMIC *if and only if any element of* P *can be expressed as a join* $\bigsqcup_i p_i$*, where all* p_i *are in* P *and are of equal size, and where there is no* q *in* P *such that* $|q| < |p_i|$*. The* p_i *are the* PSEUDOATOMS *of* P.

Pseudoatomicity characterizes a first (inclusive or exclusive) or second person lattice with, at a minimum, the [+additive] bottom element (i, u, iu) removed, or a third person lattice without, at least, its atomic stratum. We can therefore summarize the discussion with the following statement:

STATEMENT 2. *Let* P *be a join-incomplete, permutation-invariant, convex, pseudoatomic lattice region, and let* Q *be a subregion of* P. *If* Q *is permutation-invariant and convex, then* Q *is join-incomplete.*

PROOF. Let P_i be the set of pseudoatoms of P. If $\bigsqcup\mathrm{P}_i$ were in P, then convexity of P would mean that P is join-complete, contrary to assumption (since, for every pair of elements, p, q, in P, we would have $p_i \sqsubseteq p \sqcup q \sqsubseteq \bigsqcup\mathrm{P}_i$, where p_i is a pseudoatom of p). So, $\bigsqcup\mathrm{P}_i \notin \mathrm{P}$.

Every p_i is a subelement of some q in Q, or else Q is not permutation-invariant. Conversely, every q in Q is the join of some subset of P_i. So, $\bigsqcup\mathrm{Q} = \bigsqcup\mathrm{P}_i$. Given that $\mathrm{Q} \sqsubset \mathrm{P}$ and $\bigsqcup\mathrm{P}_i \notin \mathrm{P}$, it follows that $\bigsqcup\mathrm{P}_i = \bigsqcup\mathrm{Q} \notin \mathrm{Q}$. Thus, Q is join-incomplete. ■

APPENDIX C: EXPANSION OF (+ADDITIVE(−ADDITIVE(P)))

When calculating (+additive(−additive(P))), we left Q ⊏ (−additive(P)) unexpanded, for reasons of readability. I now show that this simplification in the exposition is innocent. We wish to verify the following statement:

STATEMENT 3. *Let* P *be a join-complete lattice region and let* Q′, Q *be the lattice regions introduced, respectively, by* (+additive) *and* (−additive) *in the expression* (+additive (−additive (P)))*. That is,* Q′ *is* Q′ *in Fig. 7 and* Q *corresponds to* Q_{+-}*. So,* Q′ *and* Q *characterize the set of x's such that:*

$$\frac{\neg\forall y\,(\mathrm{Q}'(y) \rightarrow \mathrm{Q}'(x \sqcup y))}{\mathrm{Q}'(x) \wedge \forall z\left(\mathrm{Q}'(z) \rightarrow \dfrac{\forall y\,(\mathrm{Q}(y) \rightarrow \mathrm{Q}(y \sqcup z))}{\mathrm{Q}(z) \wedge \mathrm{Q} \sqsubset \mathrm{P}}\right) \wedge \neg\forall z\left(\dfrac{\forall y\,(\mathrm{Q}(y) \rightarrow \mathrm{Q}(y \sqcup z))}{\mathrm{Q}(z) \wedge \mathrm{Q} \sqsubset \mathrm{P}} \rightarrow \mathrm{Q}'(z)\right)}$$

Then Q′ ⊏ Q *and* Q *is join-complete.*

COMMENT. The formula in statement 3 is 17 with Q′ ⊏ [+additive](P) expanded. We can calculate its denotation without the further complication of expanding Q ⊏ P. I assume, without comment, that all cuts produce convex, permutation-invariant regions that are either join-complete or have a join-complete complement.

PROOF. It suffices to show that, if the subordinate presuppositions, Q(z) and Q ⊏ P, are satisfied, then the desired interpretation results, and that if either subordinate presupposition fails, then no elements of the lattice are characterized. For clarity, these are laid out as separate statements.

STATEMENT 4. *Let* $\mathrm{Q} \sqsubset \mathrm{P}$ *be permutation-invariant lattice regions, with* P *join-complete, and let* Q' *be a permutation-invariant region such that:*

$$\forall z\,(\mathrm{Q}'(z) \rightarrow \forall y\,(\mathrm{Q}(y) \rightarrow \mathrm{Q}(y \sqcup z))) \wedge \neg\forall z\,(\forall y\,((\mathrm{Q}(y) \rightarrow \mathrm{Q}(y \sqcup z)) \rightarrow \mathrm{Q}'(z)))$$

where $\mathrm{Q}(z)$. *Then* $\mathrm{Q}' \sqsubset \mathrm{Q}$ *and* Q *is join-complete.*

PROOF. Suppose that $\mathrm{Q}'(z)$, for some z. By hypothesis, $\mathrm{Q}(z)$. So, $\mathrm{Q}' \sqsubseteq \mathrm{Q}$. Moreover, by the second conjunct, there is some z such that $\forall y\,(\mathrm{Q}(y) \rightarrow \mathrm{Q}(y \sqcup z)) \wedge \neg\mathrm{Q}'(z)$. Hence, $\neg\mathrm{Q}'(z)$. But given, again, by hypothesis, that $\mathrm{Q}(z)$, we have $\mathrm{Q}' \neq \mathrm{Q}$. So, $\mathrm{Q}' \sqsubset \mathrm{Q}$.

Join completeness of Q under these conditions is, perhaps, less obvious. However, given any two elements a and b in Q, we can show that $a \sqcup b$ is in Q by induction over the size of $|a \sqcup b| - |a|$. Without loss of generality, assume that a is not smaller than b. For the base case, assume that $|a \sqcup b| - |a| = 1$. Then there is an atom, c, such that $a \sqcup b = a \sqcup c$, from which it also follows that c is not in a. Pick d in Q' such that c is the only atom of d that is not in a (which is possible since $\mathrm{Q}' \sqsubset \mathrm{Q}$ and both are horizontal cuts of P). Thus, we have $\mathrm{Q}'(d)$, $\mathrm{Q}(a)$, and $\forall z\,(\mathrm{Q}'(z) \rightarrow \forall y\,(\mathrm{Q}(y) \rightarrow \mathrm{Q}(y \sqcup z)))$. So, $\mathrm{Q}(a \sqcup d)$. However, $a \sqcup ((d\backslash c) \sqcup c) = (a \sqcup (d\backslash c)) \sqcup c = a \sqcup c = a \sqcup b$. So, $\mathrm{Q}(a \sqcup b)$.

For the inductive step, we assume that $a \sqcup b$ is in Q whenever a, b are in Q and $|a \sqcup b| - |a| \leq n$, and we aim to prove that the same implication holds if $|a \sqcup b| - |a| = n + 1$. Let c be an atom such that $|(a \sqcup b)\backslash c| - |a| = n$. By the inductive hypothesis, $\mathrm{Q}((a \sqcup b)\backslash c)$. We can now reiterate the procedure from the base case. We pick d in Q' that contains c and such that $d\backslash c \sqsubseteq a$. Since $\mathrm{Q}'(d)$ and $\mathrm{Q}((a \sqcup b)\backslash c)$, we have $\mathrm{Q}(((a \sqcup b)\backslash c) \sqcup d)$. But $((a \sqcup b)\backslash c) \sqcup d = ((a \sqcup b)\backslash c) \sqcup (c \sqcup (d\backslash c)) = (((a \sqcup b)\backslash c) \sqcup c) \sqcup (d\backslash c) = (a \sqcup b) \sqcup (d\backslash c) = (a \sqcup (d\backslash c)) \sqcup b = a \sqcup b$. So, $\mathrm{Q}(a \sqcup b)$. ■

STATEMENT 5. *If either presupposition of the formula below fails to be true, then the formula fails to be true:*

$$\forall z\left(\mathrm{Q}'(z) \rightarrow \frac{\forall y\,(\mathrm{Q}(y) \rightarrow \mathrm{Q}(y \sqcup z))}{\mathrm{Q}(z) \wedge \mathrm{Q} \sqsubset \mathrm{P}}\right) \wedge \neg\forall z\left(\frac{\forall y\,(\mathrm{Q}(y) \rightarrow \mathrm{Q}(y \sqcup z))}{\mathrm{Q}(z) \wedge \mathrm{Q} \sqsubset \mathrm{P}} \rightarrow \mathrm{Q}'(z)\right)$$

PROOF. If $\mathrm{Q}(z)$ or $\mathrm{Q} \sqsubset \mathrm{P}$ fails, then the formula reduces to $\forall z\,(\mathrm{Q}'(z) \rightarrow \textit{undefined}) \wedge \neg\forall z\,(\textit{undefined} \rightarrow \mathrm{Q}'(z))$. If $\mathrm{Q}'(z)$ is true, for a given z, then this formula reduces to $(\textit{true} \rightarrow \textit{undefined}) \wedge \neg(\textit{undefined} \rightarrow \textit{true})$. This is not true (in Kleene's three-valued logic, among other systems). Similarly, if $\mathrm{Q}'(z)$ is false, for a given z, the formula reduces to $(\textit{false} \rightarrow \textit{undefined}) \wedge \neg(\textit{undefined} \rightarrow \textit{false})$. Here, the second conjunct is not true (though the first is), so again the formula is not true. Given that either z is in Q' or it is not, the formula is never true when either of its presuppositions fails. ■

COMMENT. The formula of statement 5 is a presupposition of that in statement 3. If the former is undefined, then so is the latter. Thus, the only interpretation that the formula of statement 3 can attain is that permitted by statement 4. This completes the proof of statement 3. ■

APPENDIX D: AXIOM OF EXTENSION

STATEMENT 6. $\{a, a\} = \{a\}$.

PROOF. In the standard language of the Zermelo Fraenkel theory, the axiom of extension states that $\forall A\,\forall B\,(\forall C\,(C \in A \leftrightarrow C \in B) \rightarrow A = B)$. That is, sets are identical whenever anything contained in one is contained in the other and vice versa. In particular, then, $\{a, a\} = \{a\}$ because everything in $\{a, a\}$ is in $\{a\}$ and vice versa. ■

APPENDIX E: $\neq \Leftrightarrow \sqsubset$

STATEMENT 7. *If* $\exists x\, \mathrm{P}(x)$*, then A3 (strict cumulativity; Krifka 1992:32) and A4 are equivalent.*

(A3) $\forall x\, \forall y\, ((\mathrm{P}(x) \wedge \mathrm{P}(y)) \rightarrow \mathrm{P}(x \sqcup y)) \wedge \neg\exists x\, \forall y\, (\mathrm{P}(x) \wedge (\mathrm{P}(y) \rightarrow x = y))$
(A4) $\forall x\, \forall y\, ((\mathrm{P}(x) \wedge \mathrm{P}(y)) \rightarrow \mathrm{P}(x \sqcup y)) \wedge \exists x\, \exists y\, (\mathrm{P}(x) \wedge \mathrm{P}(y) \wedge y \sqsubset x)$

PROOF. Assume that A3 is true. Then $\exists x\, \mathrm{P}(x)$ and the second conjunct, $\neg\exists x\, \forall y\, (\mathrm{P}(x) \wedge (\mathrm{P}(y) \rightarrow x = y))$, guarantee that there are $a \neq b$ such that $\mathrm{P}(a)$ and $\mathrm{P}(b)$. Given that $\forall x\, \forall y\, ((\mathrm{P}(x) \wedge \mathrm{P}(y)) \rightarrow \mathrm{P}(x \sqcup y))$, we have $\mathrm{P}(a \sqcup b)$ and, so, $\mathrm{P}(a) \wedge \mathrm{P}(a \sqcup b) \wedge a \sqsubset (a \sqcup b)$, bearing in mind that $b \neq \emptyset$.[1] Since $\{A \wedge B, C\} \vDash A \wedge C$, we have $\forall x\, \forall y\, ((\mathrm{P}(x) \wedge \mathrm{P}(y)) \rightarrow \mathrm{P}(x \sqcup y)) \wedge (\mathrm{P}(a) \wedge \mathrm{P}(a \sqcup b) \wedge a \sqsubset (a \sqcup b))$. Existential quantification over a and $a \sqcup b$ yields A4.

Conversely, assume that A4 is true. Then there are $a \sqsubset b$ such that $\mathrm{P}(a)$ and $\mathrm{P}(b)$. It follows from $a \sqsubset b$ that $a \neq b$. So, we have $\mathrm{P}(a) \wedge \mathrm{P}(b) \wedge a \neq b$. The proof concludes using the same entailment and existential quantification as for the first part. ■

COMMENT. A3 is the near equivalent of the composition [+additive] ∘ [+minimal](P) = [+additive](P) ∧ [+minimal](P). For [+minimal], the only difference is that $\mathrm{P}(x)$ is presupposed, not asserted. For [+additive], there is an additional difference, in that a free variable is introduced. The reason that A3 alone is not sufficient for the purposes of approximative number is as explained in §3.2. The ramifications (for aspect) of removing parts of strict cumulativity from the scope of negation remain to be investigated.

APPENDIX F: CONVEXITY, PERMUTATION INVARIANCE, AND JOIN COMPLETENESS

Following 32, we discussed a convergence of concepts: that permutation invariance and join completeness force convexity. Formally:

STATEMENT 8. *Any nonconvex subregion of a lattice region either fails permutation invariance or has a join-incomplete complement.*

PROOF. Let $\mathrm{Q} \sqsubset L$ be lattice regions. If Q is permutation-sensitive, then there is nothing to prove. Otherwise, we wish to show that the complement of Q is join-incomplete (which we do using the same style of proof as in the induction in Appendix C). Since Q is nonconvex, there exist x, y in Q and u not in Q such that $x \sqsubset u \sqsubset y$. For some such x, y, pick the maximal such u and then, relative to that u, pick the minimal such y in Q. Then $y = u \sqcup a$, for some atom a. Since Q is permutation-invariant, x is in Q if and only if all other elements of size $|x|$ are in Q. So, there is some v not in Q and some atom b such that $|v| = |u|$, $y = v \sqcup b$, $b \sqsubseteq u$, and $a \sqsubseteq v$. Thus, u, v are not in Q, but $u \sqcup v = (u \sqcup b) \sqcup (v \sqcup a) = (u \sqcup a) \sqcup (v \sqcup b) = y \sqcup y = y$, which is in Q. This shows that the complement of Q is join-incomplete. ■

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[1] The empty set is taken not to be a possible value for x in $\exists x\, \mathrm{P}(x)$, on the grounds that the empty set is not a cat and so the predicate cat(x), or any other predicate, cannot be true of the empty set. (The empty set obviously does satisfy the predicate empty-set(x), but, as $\emptyset$ is unique, empty-set(x) is not a relevant value of $\mathrm{P}(x)$ in the main text.)